RESEARCH STATEMENT

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As a professional mathematician, the majority of my work has been in the areas of partition theory, analytic number theory, generalized Ramsey theory, and extremal graph theory. In the statement below, I will outline some of my contributions in these areas, and try to briefly explain the significance of my work by placing it in historical context.

The study of partitions dates back over 300 years [D], and really began as an area of study with Euler [Eu]. Although it has venerable roots, partition theory still plays an important role at the cutting edge of mathematics today. First and foremost, partition theory holds some of the most beautiful results and fascinating open problems in the fields of additive number theory and enumerative combinatorics, some of which I will mention below. In addition to being an important part of combinatorics and additive number theory, there are links between partitions and Gauss' class number problem [OS], representation theory [JK], modular forms, elliptic curves [O2], modular motives [GO], and even statistical mechanics [Ba]. Many of these connections are non-obvious, but have emerged naturally as the theory of partitions has advanced. One of the questions that led to the discovery of some of these connections is, "When are the values of partition functions divisible by various integers M?"

Congruences of Partition Functions

Although the study of partition functions dates back to Euler, the first explorations of the divisibility of $p(n)$, the number of ordinary partitions n, began with Ramanujan. In [R1], after examining a table constructed by MacMahon containing the first 200 values of $p(n)$, Ramanujan conjectured the following congruences for the ordinary partition function:

If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$, then for every nonnegative integer n,

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}.$$

He then went on to prove his conjecture in the cases where a is arbitrary [R2], b is at most two, and c is at most one. Within 50 years, his conjecture was proved for all powers of 7 (after a slight correction) by Watson [Wa], and finally for all powers of 11 by Atkin [At]. These congruences for the ordinary partition function are only the edge of a much larger picture.

When I began study in this area, the congruence properties of other partition functions in arithmetic progressions had also been widely studied. Gandhi [Gan] and Newman [N1] gave several congruences for colored partition functions. In [An], Andrews proved congruences for generalized Frobenius partition functions, and both Kolitsch [Koli1]-[Koli8] and Sellers [S1]-[S3] have continued study along these lines. In [GKS], Garvan, Kim, and Stanton produce formulas for the number of t-core partitions of n for certain values of t, and these formulas imply several congruences for t-core partition functions (which play an important role in the representation theory of the symmetric group). In [HS1] and [HS2], Hirschhorn and Sellers reveal many congruence properties of the 4-core partition function.

My contribution to this area began in [EO], where Ono and I use the theory of modular forms to give a uniform algorithm for reducing the proofs of suspected congruences of $p_r(n)$ (including $p(n) = p_1(n)$), the number of partitions of n into r colors, in arithmetic progressions to a finite verification. This paper was the first of its kind in the sense that, while several techniques for proving congruences in arithmetic progressions of partition functions existed previously, there was no uniform way in which every such congruence could be proven. In [ES1], Sellers and I extend the techniques developed in [EO] to t-core partition functions, and in [ES2], Sellers and I extend the techniques developed in [EO] and [ES1] to 2-colored Frobenius partitions in order to shed light on an outstanding conjecture in that area. Ultimately, in [Ei3], I reduce the proof of any congruence in any arithmetic progression of any arithmetic function $b(n)$ that has a generating function which can be expressed in the form

$$\sum_{n=0}^{\infty} b(n)q^n = \prod_{i=1}^{M}\prod_{j=1}^{\infty}(1-q^{ij})^{e_i}$$

to a finite verification. Functions $b(n)$ with this property include all of the aforementioned partition functions, the number of partitions of n into distinct parts, the Ramanujan τ-function, the number of representations of n as a sum of k squares, and a wide variety of other classical arithmetic functions.

Using this technique to prove congruences has become a staple of modern partition theory. There have been dramatic breakthroughs regarding the divisibility properties of $p(n)$ in past few years. I will not enumerate them here, but many of these theorems and/or the examples for these results rely upon this technique of reducing the proof of a conjectured congruence to a finite verification (see [LO], [O3], and [We] for a few examples).

The Parity of $p(n)$

The parity of $p(n)$, has been studied for at least a century, yet it still remains something of a mystery. Although much work has been done, the known lower bounds for the number of even and odd values of $p(n)$ for $n \leq N$ still appear to have a great deal of room for improvement. Computing the parity of specific values of $p(n)$ has been possible since the time of Euler, but information about the overall distribution of the parity of $p(n)$ has been much more elusive. It is a long standing conjecture that $p(n)$ is odd (and even) half of the time asymptotically, but it was not until 1959 that Kolberg proved that it is both odd and even infinitely often [Kolb]. In 1967 [PS], Parkin and Shanks undertook an in-depth computational investigation of the parity of $p(n)$, and their work (including the parity of over two million values of $p(n)$) does support the conjecture. In 1996 [O1], Ono showed that $p(n)$ is even infinitely often in every arithmetic progression, and also that it is odd infinitely often in every arithmetic progression, provided it is at least once. In 1998 [NRS], Nicolas, Rusza, and Sárközy proved that

$$\#\{n \leq N : p(n) \text{ is odd}\} > \sqrt{N}e^{-(\log 2+\varepsilon)\frac{\log N}{\log\log N}},$$

and that for some constant c,

$$\#\{n \leq N : p(n) \text{ is even}\} > c\sqrt{N}$$

for N sufficiently large. Amazingly, these were the first lower bounds that were greater than a power of $\log N$. In an appendix to [NRS], Serre showed that for any t and r,

$$\lim_{N\to\infty} \frac{\#\{n \leq N : p(tn+r) \text{ is even}\}}{\sqrt{N}} = \infty.$$

In 1999 [Ah1], Ahlgren quantified Ono's work, and gave new lower bounds for the number of odd values of $p(n)$ in arithmetic progressions. The explicit bound he gives for the number of odd values of $p(n)$ is

$$\#\{n \leq N : p(n) \text{ is odd}\} > \frac{4\sqrt{N}}{\log 8N}$$

for N sufficiently large. The technology used in this result involves both theorems from the theory of modular forms and properties of ℓ-adic Galois representations. In [Ei5], I give a better (and currently the best-known) lower bound for the number of odd values of $p(n)$ using only Euler's Pentagonal Number Theorem, Jacobi's Identity, the classical formula for $r_2(n)$ (the number of representations of n as a sum of two squares), and the prime number theorem for a single arithmetic progression.

The Distribution of $p(n)$ Modulo 3

Despite a wealth of new information about $p(n)$, amazingly, the following question still remains open:

$$\text{Is } p(n) \equiv 0 \pmod 3 \text{ infinitely often?}$$

In fact, it is not known whether $p(n)$ takes on any residue modulo 3 infinitely often. This is a very shocking gap in our knowledge about $p(n)$, especially since all empirical evidence suggests that $p(n)$ is uniformly distributed modulo 3. Yet, almost nothing about the distribution of $p(n)$ modulo 3 is known.

In 1960, Newman [N2] showed that $p(n)$ is not ultimately periodic modulo any M, including 3. In 1999, Ahlgren [Ah2], using the theory of modular forms and deep results of Deligne, Serre, and Shimura, proved a general theorem that has as a special case that $\#\{n \leq X |\ p(n) \not\equiv 0 \pmod 3\} \gg \sqrt{X}/\log X$ (in fact, he showed that this also holds in any arithmetic progression). Until very recently, the two results just mentioned comprised the only known information about $p(n)$ modulo 3 (excepting, of course, our ability to compute any particular value of $p(n)$). In [Ei4], I show that for $r = 1$ or 2, $\#\{n \leq X : p(n) \not\equiv r \pmod 3\} \gg \sqrt{X}$. This is noteworthy because of the complete lack of other information about $p(n)$ modulo 3, but also because the proof does not require the use of modular forms, and as a result, vastly more general theorems are true. For example, we can extend this result from the modulus 3 to any modulus M, from $p(n)$ to an enormous class of arithmetic functions that include many classical partition functions, and from arbitrary n to n in certain unions of arithmetic progressions.

Applying Combinatorial Methods to Partition Theory

The conventional wisdom in the theory of partitions is that the combinatorial approach to proving identities is often the most difficult, but yields the most insight. In many cases, giving combinatorial proofs to known results has resulted in vast generalizations of the original theorems due to this additional insight. A prime example appears in the study of partitions via the gaps between their parts.

In [B], Bowman introduces the notion of "partitions with parts in the gaps", and he proves several identities between classical partition functions and partition functions of this new type. His methods involve manipulations of generating functions that can be expressed as continued fractions. In [Ei1], I give purely combinatorial proofs of some of Bowman's results, and these new proofs make several generalizations come to light in a completely natural way. For example, if we let $p_{k,m}(j,n)$ be the number of partitions of n into j parts where each part is $\equiv k \pmod{m}, 1 \leq k \leq m$, and we let $p^*_{k,m}(j,n)$ be the number of partitions of n into j parts where each part is $\equiv k \pmod{m}$ with "parts of size k in the gaps", then $p^*_{k,m}(j,n) = p_{k,m}(j,n)$. This result is more or less a three parameter generalization of what had appeared previously. Also, in [Ei1], there are further generalizations of the above result in several directions.

A key point here is that once a small sample of identities were known about partitions with parts in the gaps, giving *combinatorial* proofs of these identities led to a wide variety of generalizations and a deeper understanding. This theme permeates many of my publications, and will likely continue to play an important role in my current and future research projects (see [Ei2], [EH], [EO'B] for further examples).

Problems in Extremal Graph Theory

Working on problems in extremal graph theory started as a hobby, but it has now grown into a full-blown research interest. I have now co-authored two papers in extremal graph theory, and my graph-theoretic work has contributed to and been cited in several other publications.

My first contributions to extremal graph theory came in the form of providing best-known constructions of one sort or another. These contributions are mentioned several places in [M2], and also in [AFM] and [JMSW].

In [EMOW], Mubayi, O'Bryant, West and I solve an edge-labeling problem on theta-graphs. If we define an *edge-labeling* f of a graph G to be an injection from $E(G)$ to the integers, then the *edge-bandwidth* of G is $B'(G) = \min_f\{B'(f)\}$, where $B'(f)$ is the maximum difference between labels of two incident edges of G. The *m-theta graph* $\Theta(l_1, \dots, l_m)$ is the graph consisting of m pairwise internally disjoint paths with common endpoints and lengths $l_1 \leq \cdots \leq l_m$. This class of graphs is especially interesting for studying edge-bandwidth because none of the standard techniques for obtaining lower bounds applies, and the optimal constructions are far from obvious. The end result of [EMOW] is the exact determination of the edge-bandwidth of all m-theta graphs.

In [EM], Mubayi and I use an ingenious construction from [M1] to make substantial advances on a generalization of the Ramsey problem for multicolorings. To be precise, if we define a *(p,q)-coloring of K_n* to be a coloring of $E(K_n)$ in which the edges of every copy of $K_p \subseteq K_n$ together receive at least q colors, and $f(n,p,q)$ to be the minimum number of colors in a (p,q)-coloring of K_n (see [Er]), then we show that as $n \to \infty$,

$$f(n,p,q) < e^{\sqrt{4\log 2 \log n}\ (1+o(1))}$$

for a particular infinite family of pairs (p,q). The previous best known bounds for each of the pairs (except the first) in this particular infinite family was a fixed power of n (see [EG]).

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