Solution to the Midterm

Problem 1. Let C be a binary block code defined by the following generator matrix

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Find its Parity-check matrix H.

The systematic generator matrix G is not in standard from; to obtain a generator matrix in standard form, first we define a permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which transforms the generator matrix

$$G = [\, e_4 \quad e_5 \quad e_2 \quad G_4 \quad G_5 \quad e_3 \quad G_7 \quad G_8 \quad e_1 \,] \qquad \text{into}$$
$$G_1 = G * P = [\, e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad G_6 \quad G_7 \quad G_8 \quad G_9 \,]$$

in standard form. Thus

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} = [\, I_5 \quad B \,].$$

The Parity-check matrix H_1 associated with G_1 is:

$$H_1 = \begin{bmatrix} B \\ I_4 \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By multiplying P by H_1, we obtain the parity matrix H, associated with G:

$$P * H_1 = H = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ H_6 \\ H_7 \\ H_8 \\ H_9 \end{bmatrix}.$$

(b) A message was encode using G. Decode the following received words:

$$W = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

In order to decode W, we need to find the syndrome matrix:

$$S = W * H = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_7 \\ \theta \\ H_6 \\ H_8 \\ H_5 \\ \theta \\ H_3 \\ H_9 \end{bmatrix}.$$

We conclude that the error matrix is:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence the decoded words are:

$$V = W + E = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

By Choosing the first five columns of the matrix $M = V * P$ and assigning corresponding letters to each row, we obtain:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{C} \\ \mathcal{A} \\ \mathcal{L} \\ \mathcal{L} \\ \\ \mathcal{H} \\ \mathcal{O} \\ \mathcal{M} \\ \mathcal{E} \end{bmatrix}.$$

Problem 2. Let $S = \{11001100, 10000111, 10000111, 00110011, 10110100\}$ be a subset of $\mathbb{K}^8$ generating the linear binary block code C.

a. Find a generator matrix G.

Using the words in S, first we define the matrix M in order to find a generator matrix G for the code generated by the words in S.

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Notice that, the second row and the third row are identical and are equal to the sum of the last two rows. By deleting both, the second and third rows, we obtain a 3×8 generator matrix of rank three:

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

b. Use row operations, to obtain a generator matrix G_1 in standard form.

By using the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; E2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; E3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$
$$E4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad E5 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

we obtain:

$$G_1 = E_5\, E_4\, E_3\, E_2\, E_1\, G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

c. Find a parity-check matrix associated with G_1.

$$H_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

d. Is C a self-dual code?

Although G_1 is orthogonal to G_1^t (i.e., $G_1 * G_1^t$ is a 3×3 zero matrix), the Code C is not self dual, because the 8×3 matrix G_1^t could not be a parity-check matrix for G_1, since any parity-check matrix for the code C must be an 8×5 matrix.

e. Which one of the following words are codewords? Explain all the reasons.

$$w_1 = 00011110, \quad w_2 = 10011000, \quad w_3 = 11110100, \quad w_4 = 11110011$$

Define

$$W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The fact that the syndrome

$$S = W * H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

has no zero row implies that none of those words are codewords. It should be noted that since the third row of S is the same as the second row of H_1, the word w_3 could be corrected by changing its second digit from zero to one.

Problem 3. First define a [7, 4, 3] Hamming code

$$G = [\, G_1 \quad G_2 \quad G_3 \quad G_4 \quad G_5 \quad G_6 \quad G_7 \,],$$

where $G_1,\ G_2,$ and G_4 are parity bits and $G_3 = e_3$, $G_5 = e_2$, $G_6 = e_1$, and $G_7 = e_4$. Then decode the following received words using the Hamming algorithm:

$$w_0 = 0001111 \quad w_1 = 0011011 \quad w_2 = 0011101 \quad w_3 = 1001100 \quad w_4 = 1000011$$

Here is a systematic generating matrix of the [7, 4, 3] Hamming code using the above algorithm:

$$G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{array}{rlllll} \textit{Received}: & w_0 = 0001111 & w_1 = 0011011 & w_2 = 0011101 & w_3 = 1001100 & w_4 = 1000011 \\ (\alpha\,\beta\,\gamma): & (0\,0\,0) \longleftrightarrow 0 & (0\,1\,1) \longleftrightarrow 6 & (1\,0\,1) \longleftrightarrow 5 & (0\,0\,0) \longleftrightarrow 0 & (0\,0\,0) \longleftrightarrow 0 \\ \textit{Decoded}: & v_0' = 0001111 & v_1' = 0011001 & v_2' = 0011001 & v_3' = 1001100 & v_4' = 1000011 \end{array}$$

Problem 4. The sequence

$$39 \quad 48 \quad 29 \quad 34 \quad 55 \quad 73 \quad 3 \quad 4 \quad 33 \quad 40 \quad 11 \quad 13 \quad 21 \quad 22$$

encoding an important message by using the matrix $A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ have been received. Decode the message, assuming that the blank is zero and

$$A = 1,\ B = 2,\ C = 3,\ \ldots,\ O = 15,\ P = 16,\ Q = 17,\ R = 18, \ldots, y = 25,\ Z = 26.$$

First, we need to find $A^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$. Then, we change the sequence into a 2×7 matrix

$$M = \begin{pmatrix} 39 & 48 & 29 & 34 & 55 & 73 & 3 \\ 4 & 33 & 40 & 11 & 13 & 21 & 22 \end{pmatrix}.$$

To decode those numbers and obtain the sent message, we need to find

$$A^{-1}M = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \begin{bmatrix} 39 & 48 & 29 & 34 & 55 & 73 & 3 \\ 4 & 33 & 40 & 11 & 13 & 21 & 22 \end{bmatrix} = \begin{bmatrix} 12 & 9 & 14 & 5 & 1 & 18 & 0 \\ 1 & 12 & 7 & 5 & 2 & 18 & 1 \end{bmatrix}.$$

By assigning letters to those numbers, we obtain the message:

$$\mathcal{L}\,\mathcal{I}\,\mathcal{N}\,\mathcal{E}\,\mathcal{A}\,\mathcal{R} \quad \mathcal{A}\,\mathcal{L}\,\mathcal{G}\,\mathcal{E}\,\mathcal{B}\,\mathcal{R}\,\mathcal{A}$$

Problem 5. The following $\mathcal{ISBN}$: $\mathbf{2-751-433}\blacksquare\mathbf{4-A}$ has been received with smudge. What is the missing digit?

First we assign x to $\blacksquare$ and then we use the following procedure:

$$\begin{array}{cccccccccc} & 2 & 7 & 5 & 1 & 4 & 3 & 3 & x & 4 \\ \times & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ = & 2 & 14 & 15 & 4 & 20 & 18 & 21 & 8x & 36 \end{array}$$

We evaluate:

$$2+14+15+4+20+18+21+8x+36 \equiv 9+8x \quad mod\ 11.$$

In a system of base 11, A is assign to 10. Thus

$$\begin{aligned} &9+8x \equiv A = 10 \quad mod\ 11 \Longrightarrow 8x \equiv 1 \quad mod\ 11; \\ &8\times 1 \equiv 8 \neq 1 \quad mod\ 11 \quad 8\times 2 \equiv 5 \neq 1 \quad mod\ 11 \\ &8\times 3 \equiv 2 \neq 1 \quad mod\ 11 \quad 8\times 4 \equiv 10 \neq 1 \quad mod\ 11 \\ &8\times 5 \equiv 7 \neq 1 \quad mod\ 11 \quad 8\times 6 \equiv 4 \neq 1 \quad mod\ 11\,. \end{aligned}$$

Finally

$$8\times 7 \equiv 1 \quad mod\ 11 \Longrightarrow x = 7.$$