**Characteristic Polynomial**

**Preliminary Results.** Let $A = (a_{ij})$ be an $n \times n$ matrix. If $Au = \lambda u$, then $\lambda$ and $u$ are called the *eigenvalue* and *eigenvector* of $A$, respectively. The eigenvalues of $A$ are the roots of the characteristic polynomial

$$K_A(\lambda) = \det(\lambda I_n - A).$$

The eigenvectors are the solutions to the *Homogeneous system*

$$(\lambda I_n - A)X = \theta.$$

Note that $K_A(\lambda)$ is a *monic polynomial* (i.e., the leading coefficient is one).

**Cayley-Hamilton Theorem.** If $K_A(\lambda) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ is the characteristic polynomial of the $n \times n$ matrix $A$, then

$$K_A(A) = A^n + p_1A^{n-1} + \cdots + p_{n-1}A + p_nI_n = Z_n,$$

where $Z_n$ is the $n \times n$ zero matrix.

**Corollary.** Let $K_A(\lambda) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ be the characteristic polynomial of the $n \times n$ invertible matrix $A$. Then

$$A^{-1} = \frac{1}{-p_n} [A^{n-1} + p_1A^{n-2} + \cdots + p_{n-2}A + p_{n-1}I_n].$$

**Proof.** According to the Cayley Hamilton’s theorem we have

$$A [A^{n-1} + p_1A^{n-2} + \cdots + p_{n-1}I_n] = -p_nI_n,$$

Since $A$ is nonsingular, $p_n = (-1)^n \det(A) \neq 0$; thus the result follows.

**Newton’s Identity.** Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the polynomial

$$P(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n.$$

If $s_k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$, then

$$c_k = -\frac{1}{k} (s_k + s_{k-1}c_1 + s_{k-2}c_2 + \cdots + s_2c_{k-2}c_1 + s_1c_{k-1}).$$

**Proof.** From

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_{n-1})(\lambda - \lambda_n)$$

and the use of logarithmic differentiation, we obtain

$$\frac{P'(\lambda)}{P(\lambda)} = \frac{n\lambda^{n-1} + (n-1)c_1\lambda^{n-2} + \cdots + 2c_{n-2}\lambda + c_{n-1}}{\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n} = \sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i}.$$ 

By using the geometric series for $\frac{1}{\lambda - \lambda_i}$ and choosing $|\lambda| > \max_{1 \leq i \leq n} |\lambda_i|$, we obtain

$$\sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i} = \frac{n}{\lambda} + \frac{s_1}{\lambda^2} + \frac{s_2}{\lambda^3} + \cdots.$$

Hence

$$n\lambda^{n-1} + (n-1)c_1\lambda^{n-2} + \cdots + c_{n-1} = (\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n) \left(\frac{n}{\lambda} + \frac{s_1}{\lambda^2} + \frac{s_2}{\lambda^3} + \cdots\right).$$

By equating both sides of the above equality we may obtain the Newton’s identities.
The Method of Direct Expansion

The characteristic polynomial of an \( n \times n \) matrix \( A = (a_{ij}) \) is defined as:

\[
K_A(\lambda) = \det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = \lambda^n - \sigma_1 \lambda^{n-1} + \sigma_2 \lambda^{n-2} - \cdots + (-1)^n \sigma_n,
\]

where

\[
\sigma_1 = \sum_{i=1}^{n} a_{ii} = \text{trace}(A)
\]

is the sum of all first-order diagonal minors of \( A \),

\[
\sigma_2 = \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}
\]

is the sum of all second-order diagonal minors of \( A \),

\[
\sigma_3 = \sum_{i<j<k} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}
\]

is the sum of all third-order diagonal minors of \( A \), and so forth. Finally,

\[
\sigma_n = \det(A)
\]

There are \( \binom{n}{k} \) diagonal minors of order \( k \) in \( A \). From this we find that the direct computation of the coefficients of the characteristic polynomial of an \( n \times n \) matrix is equivalent to computing

\[
\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1
\]

determinants of various orders, which, generally speaking, is a major task. This has given rise to special methods for expanding characteristic polynomial. We shall explain some of these methods.

Example. Compute the characteristic polynomial of \( A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -4 \\ 1 & 0 & 2 \end{pmatrix} \).

We have:

\[
\sigma_1 = 1 + 1 + 2 = 4, \quad \sigma_2 = \frac{1}{2} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = (-3) + (2) + (-1) = -2,
\]

and

\[
\sigma_3 = \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -17.
\]

Thus

\[
K_A(\lambda) = \det(\lambda I_3 - A) = \lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = \lambda^3 - 4\lambda - 2\lambda + 17.
\]
\textbf{Leverrier's Algorithm.} This method allows us to find the characteristic polynomial of any $n \times n$ matrix $A$ using the trace of the matrix $A^k$, where $k = 1, 2, \ldots, n$. Let
\[ \sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \]
be the set of all eigenvalues of $A$ which is also called the spectrum of $A$. Note that
\[ s_k = \text{trace}(A^k) = \sum_{i=1}^{n} \lambda_i^k, \text{ for all } k = 1, 2, \cdots, n. \]

Let
\[ K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_n - 1 + p_n \]
be the characteristic polynomial of the matrix $A$, then for $k \leq n$, the \textit{Newton's identities} hold true:
\[ p_k = -\frac{1}{k} [s_k + p_1 s_{k-1} + \cdots + p_{k-1} s_1] \quad (k = 1, 2, \cdots, n) \]

\textbf{Example.} Let $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix}$. Then
\[
A^2 = \begin{pmatrix} 1 & 8 & 4 & 0 \\ 9 & -1 & -1 & 9 \\ 13 & -12 & 5 & 8 \\ 15 & -12 & -6 & 7 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 17 & 6 & 13 & 19 \\ 42 & -28 & 8 & 23 \\ 43 & -9 & -16 & 22 \\ 19 & -11 & -3 & -17 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 125 & -48 & 16 & 104 \\ 122 & -23 & -22 & 46 \\ 90 & -40 & 41 & -12 \\ -66 & 120 & 0 & -107 \end{pmatrix}.
\]
So $s_1 = 4$, $s_2 = 12$, $s_3 = -44$, and $s_4 = 36$. Hence
\[
\begin{align*}
p_1 &= -s_1 = -4, \\
p_2 &= -\frac{1}{2}(s_2 + p_1 s_1) = -\frac{1}{2}(12 + (-4)4) = 2, \\
p_3 &= -\frac{1}{3}(s_3 + p_1 s_2 + p_2 s_1) = -\frac{1}{3}(-44 + (-4)12 + 2(4)) = 28, \\
p_4 &= -\frac{1}{4}(s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1) = -\frac{1}{4}(36 + (-4)(-44) + 2(12) + 28(4)) = -87.
\end{align*}
\]
Therefore
\[ K_A(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 28\lambda - 87 \]
and
\[ A^{-1} = \frac{1}{87} \cdot [A^3 - 4A^2 + 2A + 28I_4] = \]
\[
\frac{1}{87} \begin{pmatrix} 17 & 6 & 13 & 19 \\ 42 & -28 & 8 & 23 \\ 43 & -9 & -16 & 22 \\ 19 & -11 & -3 & -17 \end{pmatrix} - 4 \begin{pmatrix} 1 & 8 & 4 & 0 \\ 9 & -1 & -1 & 9 \\ 13 & -12 & 5 & 8 \\ 15 & -12 & -6 & 7 \end{pmatrix} + 2 \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix} + 28 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
\[ A^{-1} = \frac{1}{87} \begin{pmatrix} 43 & -22 & -1 & 17 \\ 8 & 4 & 16 & -11 \\ -5 & 41 & -10 & -4 \\ -33 & 27 & 21 & -9 \end{pmatrix}. \]
\section*{The Method of Souriau (or Fadeev and Frame)} This is an elegant modification of the Leverrier’s method.

Let $A$ be an $n \times n$ matrix, then define
\begin{align*}
A_1 &= A, & q_1 &= -\text{trace}(A_1), & B_1 &= A_1 + q_1 I_n \\
A_2 &= AB_1, & q_2 &= -\frac{1}{2}\text{trace}(A_2), & B_2 &= A_2 + q_2 I_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_n &= AB_{n-1}, & q_n &= -\frac{1}{n}\text{trace}(A_n), & B_n &= A_n + q_n I_n
\end{align*}

**Theorem.** $B_n = Z_n$, and
\[ K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + q_1 \lambda^{n-1} + \cdots + q_{n-1} \lambda + q_n. \]

If $A$ is nonsingular, then
\[ A^{-1} = -\frac{1}{q_n} B_{n-1}. \]

**Proof.** Suppose the characteristic polynomial of $A$ is
\[ K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1} \lambda + p_n, \]
where $p_k$’s are defined in the Leverrier’s method.

Clearly $p_1 = -\text{trace}(A) = -\text{trace}(A_1) = q_1$, and now suppose that we have proved that
\[ q_1 = p_1, q_2 = p_2, \ldots, q_{k-1} = p_{k-1}. \]

Then by the hypothesis we have
\begin{align*}
A_k &= AB_{k-1} = A(A_k - q_k I_n) = AA_{k-1} + q_k A \\
&= A[A(A_k - q_k I_n)] + q_k A \\
&= A^2 A_{k-1} + q_k A^2 + q_k A \\
&= \vdots \vdots \vdots \vdots \vdots \vdots \\
&= A^k + q_1 A^{k-1} + \cdots + q_{k-1} A.
\end{align*}

Let $s_i = \text{trace}(A^i)$ ($i = 1, 2, \ldots, k$), then by Newton’s identities
\begin{align*}
-k q_k &= \text{trace}(A_k) = \text{trace}(A^k) + q_1 \text{trace}(A^{k-1}) + \cdots + q_{k-1} \text{trace}(A) \\
&= s_k + q_1 s_{k-1} + \cdots + q_{k-1} s_1 \\
&= s_k + p_1 s_{k-1} + \cdots + p_{k-1} s_1 \\
&= -k p_k.
\end{align*}

showing that $p_k = q_k$. Hence this relation holds for all $k$.

By the Cayley-Hamilton theorem,
\[ B_n = A^n + q_1 A^{n-1} + \cdots + q_{n-1} A + q_n I_n = Z_n. \]

and so
\[ B_n = A_n + q_n I_n = Z_n; \quad A_n = AB_{n-1} = -q_n I_n. \]

If $A$ is nonsingular, then $\det(A) = (-1)^n K_A(0) = (-1)^n q_n \neq 0$, and thus
\[ A^{-1} = -\frac{1}{q_n} B_{n-1}. \]
\[ \tag*{\textit{California State University, East Bay}} \]
Example. Find the characteristic polynomial and if possible the inverse of the matrix

\[
A = \begin{pmatrix}
1 & 2 & 1 & -1 \\
1 & 0 & 2 & 1 \\
2 & 1 & -1 & 3 \\
4 & -5 & 0 & 4 \\
\end{pmatrix}.
\]

For \( k = 1, 2, 3, 4 \), compute

\[
A_k = AB_{k-1} \quad q_k = \frac{-1}{k} \text{trace}(A_k), \quad B_k = A_k + q_k I_4.
\]

\[
A_1 = \begin{pmatrix}
1 & 2 & 1 & -1 \\
1 & 0 & 2 & 1 \\
2 & 1 & -1 & 3 \\
4 & -5 & 0 & 4 \\
\end{pmatrix}, \quad q_1 = -4, \quad B_1 = \begin{pmatrix}
-3 & 2 & 1 & -1 \\
1 & -4 & 2 & 1 \\
2 & 1 & -5 & 3 \\
4 & -5 & 0 & 0 \\
\end{pmatrix};
\]

\[
A_2 = \begin{pmatrix}
-3 & 0 & 0 & 4 \\
5 & -1 & -9 & 5 \\
5 & -16 & 9 & -4 \\
-1 & 8 & -6 & -9 \\
\end{pmatrix}, \quad q_2 = 2, \quad B_2 = \begin{pmatrix}
-1 & 0 & 0 & 4 \\
5 & 1 & -9 & 5 \\
5 & -16 & 11 & -4 \\
-1 & 8 & -6 & -7 \\
\end{pmatrix};
\]

\[
A_3 = \begin{pmatrix}
15 & -22 & -1 & 17 \\
8 & -24 & 16 & -11 \\
-5 & 41 & -38 & -4 \\
-33 & 27 & 21 & -37 \\
\end{pmatrix}, \quad q_3 = 28, \quad B_3 = \begin{pmatrix}
43 & -22 & -1 & 17 \\
8 & 16 & -11 \\
-5 & 41 & -10 & -4 \\
-33 & 27 & 21 & -9 \\
\end{pmatrix};
\]

\[
A_4 = \begin{pmatrix}
87 & 0 & 0 & 0 \\
0 & 87 & 0 & 0 \\
0 & 0 & 87 & 0 \\
0 & 0 & 0 & 87 \\
\end{pmatrix}, \quad q_4 = -87, \quad B_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Therefore the characteristic polynomial of \( A \) is:

\[
K_A(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 28\lambda - 87.
\]

Note that \( A_4 \) is a diagonal matrix, so we only need to multiply the first row of \( A \) by the first column of \( B_3 \) to obtain 87. Since \( q_4 = -87 \), the matrix \( A \) has an inverse.

\[
A^{-1} = \frac{-1}{q_4} B_3 = \frac{1}{87} \begin{pmatrix}
43 & -22 & -1 & 17 \\
8 & 4 & 16 & -11 \\
-5 & 41 & -10 & -4 \\
-33 & 27 & 21 & -9 \\
\end{pmatrix}.
\]

Matlab Program

\[
A = \text{input}('Enter a square matrix : ')
\]

\[
m = \text{size}(A); \quad n = m(1); \quad q = \text{zeros}(1, n); \quad B = A; \quad AB = A; \quad In = \text{eye}(n);
\]

\[
\text{for } k = 1 : n - 1, \quad q(k) = -(1/k) * \text{trace}(AB)B = AB + q(k) * In; \quad AB = A * B; \quad \text{end}
\]

\[
C = B; \quad q(n) = -(1/n) * \text{trace}(AB); \quad Q = [1 \ q];
\]

\[
\text{disp('The Characteristic polynomial looks like : ')}
\]

\[
\text{disp('K}_A(x) = x \wedge n + q(1)x \wedge (n - 1) + \ldots + q(n - 1)x + q(n))', \text{disp(' ')}},
\]

\[
\text{disp('The coefficients list c(k) is : ')}, \text{disp(' ')}},
\]

\[
\text{disp(Q), disp(' ')}
\]

\[
\text{if q(n) == 0, disp('The matrix is singular ')};
\]

\[
\text{else, disp('The matrix has an inverse. ')}}, \text{disp(' ')}
\]

\[
C = -(1/q(n)) * B;
\]

\[
\text{disp('The inverse of A is : ')}, \text{disp(' ')}},
\]

\[
\text{disp(C)}
\]

\[
\text{end}
\]

California State University, East Bay
\textbf{The Method of Undetermined Coefficients.} If one has to expand large numbers of characteristic polynomials of the same order, then the method of undetermined coefficients may be used to produce characteristic polynomials of those matrices.

Let \( A \) be an \( n \times n \) matrix and 
\[
K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n
\]
be its characteristic polynomial. In order to find the coefficients \( p_i \)'s of \( K_A(\lambda) \) we evaluate 
\[
D_j = K_A(j) = \det(jI_n - A) \quad j = 0, 1, \ldots, n - 1
\]
and obtain the following system of linear equations:
\[
\begin{align*}
1^n + p_11^{n-1} + \cdots + p_n &= D_0 \\
2^n + p_12^{n-1} + \cdots + p_n &= D_1 \\
&\vdots \\
(n - 1)^n + p_1(n - 1)^{n-1} + \cdots + p_n &= D_{n-1}
\end{align*}
\]
Which can be changed into:
\[
S_{n-1}P = \begin{bmatrix}
1^{n-1} & 1^{n-2} & \cdots & 1 \\
2^{n-1} & 2^{n-2} & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
(n - 1)^{n-1} & (n - 1)^{n-2} & \cdots & n - 1
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
\vdots \\
p_{n-1}
\end{bmatrix} = \begin{bmatrix}
D_1 - D_0 - 1^n \\
D_2 - D_0 - 2^n \\
\vdots \\
D_{n-1} - D_0 - (n - 1)^n
\end{bmatrix} = D.
\]
The system may be solved as follows:
\[
P = S_{n-1}^{-1}D.
\]
Since the \((n - 1) \times (n - 1)\) matrix \( S_n \) depends only on the order of \( A \), we may store \( R_n \), the inverse of \( S_{n-1} \) beforehand and use it to find the coefficients of characteristic polynomial of various \( n \times n \) matrices.

\textbf{Examples.} Compute the characteristic polynomials of the \(4 \times 4\) matrices
\[
A = \begin{pmatrix} 1 & 3 & 0 & 4 \\ 2 & -3 & 1 & 3 \\ 1 & 2 & 1 & 2 \\ -1 & 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix}.
\]
First we find
\[
S = \begin{pmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{pmatrix} \quad \text{and} \quad R = S^{-1} = -\frac{1}{12} \begin{pmatrix} -6 & 6 & -2 \\ 30 & -24 & 6 \\ -36 & 18 & -4 \end{pmatrix}.
\]
Then for the matrix \( A \) we obtain
\[
D_0 = \det(-A) = -48, \quad D_1 = \det(I_4 - A) = -72, \quad D_2 = \det(2I_4 - A) = -128 \quad \text{and} \quad D_3 = \det(3I_4 - A) = -180
\]
\[
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\]
Thus

\[ D = \begin{pmatrix} D_1 - D_0 - 1^4 \\ D_2 - D_0 - 2^4 \\ D_3 - D_0 - 3^4 \end{pmatrix} = \begin{pmatrix} -25 \\ -96 \\ -213 \end{pmatrix}. \]

Hence

\[ P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = -\frac{1}{12} \begin{pmatrix} -6 & 6 & -2 \\ 30 & -24 & 6 \\ -36 & 18 & -4 \end{pmatrix} \begin{pmatrix} -25 \\ -96 \\ -213 \end{pmatrix} = \begin{pmatrix} 0 \\ -23 \end{pmatrix}. \]

Thus

\[ A = \lambda^4 - 23\lambda^2 - 2\lambda - 48 \]

For the matrix \( B \) we have

\[ D_0 = \det(-B) = -87, \quad D_1 = \det(I_4 - B) = -60, \]
\[ D_2 = \det(2I_4 - B) = -39 \quad \text{and} \quad D_3 = \det(3I_4 - B) = -12 \]

\[ D = \begin{pmatrix} D_1 - D_0 - 1^4 \\ D_2 - D_0 - 2^4 \\ D_3 - D_0 - 3^4 \end{pmatrix} = \begin{pmatrix} 26 \\ 32 \\ -6 \end{pmatrix}. \]

Hence

\[ P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = -\frac{1}{12} \begin{pmatrix} -6 & 6 & -2 \\ 30 & -24 & 6 \\ -36 & 18 & -4 \end{pmatrix} \begin{pmatrix} 26 \\ 32 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ -28 \end{pmatrix}. \]

Thus

\[ B = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 28\lambda - 87 \]

Matlab Program

\( N = \text{input('Enter the size of your square matrix : '}\); \)
\( n = N - 1; \) \( \text{In = eye}(N); \) \( S = \text{zeros}(n); \) \( R = \text{zeros}(1, n); \)
\( \text{DSP1} = [' ' \text{for any} ' \text{, int2str(N)}, ' \text{--square matrix, you need} S \text{ ='}]; \)
\( \text{DSP2} = [' \text{Do you want to try with another} ' \text{, int2str(N)}, ' \text{--square matrix? (Yes = 1/No = 0)}']; \)
\( \% \text{DEFINING} S \)
\( \text{for} i = 1 : n; \) \( \text{for} j = 1 : n; \) \( S(i, j) = i \land (N - j); \) \text{end;} \)
\( \text{disp(' ')} \) \( \text{disp(DSP1)}, \) \( \text{disp(' ')} \) \( \text{disp(S)}, \)
\[ R = \text{inv}(S); \]
\( \text{ok} = 1; \)
\( \text{while} \) \( \text{ok} = = 1; \)
\( A = \text{input(['Enteran' , 'int2str(N)' , ' matrix A : ']);} \)
\( D0 = \text{det}(A); \)
\( \text{for} k = 1 : n; \) \( \text{D(k) = det(k * In - A);} \) \text{end;} \)
\( \text{for} i = 1 : n; \) \( \text{DD(i) = D(i) - D0 - i} \land N; \) \text{end;} \)
\( P = R \ast DD'; \)
\( \text{disp('The Characteristic polynomial looks like : ')} \)
\( \text{disp('KA(x) = x} \land n + p(1)x \land (n - 1) + ... + p(n - 1)x + p(n)' ), \) \( \text{disp(' ')} \)
\( \text{disp('The coefficients list p(k) is : '}, \) \( \text{disp(', ')} \)
\( \text{disp([1 P' D0]),} \) \( \text{disp(', ')} \)
\( \text{disp(DSP2)}, \) \( \text{disp(', ')} \)
\( \text{ok} = \text{input(DSP2)}; \)
\text{end}
Consider an \( n \times n \) matrix \( A \) and let

\[
K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1} \lambda + p_n
\]

be its characteristic polynomial. Then the \textit{companion matrix} of \( K_A(\lambda) \)

\[
F[A] = \begin{pmatrix}
-p_1 & -p_2 & -p_3 & \cdots & -p_{n-1} & -p_n \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]

is similar to \( A \) and is called the \textit{Frobenius form} of \( A \).

The method of Danilevsky (1937) applies the Gauss-Jordan method to obtain the Frobenius form of an \( n \times n \) matrix. According to this method the transition from the matrix \( A \) to \( F[A] \) is done by means of \( n - 1 \) similarity transformations which successively transform the rows of \( A \), beginning with the last, into corresponding rows of \( F[A] \).

Let us illustrate the beginning of the process. Our purpose is to carry the \( n \)th row of

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \\
\end{pmatrix}
\]

into the row \((0 \ 0 \ \cdots \ 1 \ 0)\). Assuming that \( a_{n,n-1} \neq 0 \), we replace the \((n-1)\)th row of the \( n \times n \) identity matrix with the \( n \)th row of \( A \) and obtained the matrix

\[
U_{n-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\]

The inverse of \( U_{n-1} \) is

\[
V_{n-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
v_{n-1,1} & v_{n-1,2} & v_{n-1,3} & \cdots & v_{n-1,n-1} & v_{n-1,n} \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\]

where

\[
v_{n-1,i} = -\frac{a_{ni}}{a_{n,n-1}} \quad \text{for} \quad i \neq n-1
\]

and

\[
v_{n-1,n-1} = -\frac{1}{a_{n,n-1}}.
\]
Multiplying the right side of $A$ by $V_{n-1}$, we obtain

$$AV_{n-1} = B = \begin{pmatrix}
    b_{11} & b_{12} & b_{13} & \ldots & b_{1,n-1} & b_{1n} \\
    b_{21} & b_{22} & b_{23} & \ldots & b_{2,n-1} & b_{2n} \\
    \vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\
    b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \ldots & b_{n-1,n-1} & b_{n-1,n-1}
\end{pmatrix}$$

However the matrix $B = AM_{n-1}$ is not similar to $A$. To have a similarity transformation, it is necessary to multiply the left side of $B$ by $U_{n-1} = V_{n-1}^{-1}$. Let $C = U_{n-1}AV_{n-1}$, then $C$ is similar to $A$ and is of the form

$$C = \begin{pmatrix}
    c_{11} & c_{12} & c_{13} & \ldots & c_{1,n-1} & c_{1n} \\
    b_{21} & b_{22} & b_{23} & \ldots & b_{2,n-1} & b_{2n} \\
    \vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\
    c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \ldots & c_{n-1,n-1} & c_{n-1,n-1}
\end{pmatrix}$$

Now, if $c_{n-1,n-1} \neq 0$, then similar operations are performed on matrix $C$ by taking its $(n-2)th$ row as the principal one. We then obtain the matrix

$$D = U_{n-2}CV_{n-2} = U_{n-2}U_{n-1}AV_{n-1}V_{n-2}$$

with two reduced rows. We continue the same way until we finally obtain the Frobenius form

$$F[A] = U_1U_2\cdots U_{n-2}U_{n-1}AV_{n-1}V_{n-2}\cdots V_2V_1$$

if, of course, all the $n-1$ intermediate transformations are possible.

**Exceptional case in the Danilevsky method.** Suppose that in the transformation of the matrix $A$ into its Frobenius form $F[A]$ we arrived, after a few steps, at a matrix of the form

$$R = \begin{pmatrix}
    r_{11} & r_{12} & \ldots & r_{1,k-1} & r_{1,k} & \ldots & r_{1,n-1} & r_{1n} \\
    r_{21} & r_{22} & \ldots & r_{2,k-1} & r_{2,k} & \ldots & r_{2,n-1} & r_{2n} \\
    \vdots & \vdots & \ddots & \ldots & \vdots & \ddots & \vdots & \vdots \\
    r_{k1} & r_{k2} & \ldots & r_{k,k-1} & r_{kk} & \ldots & r_{kn} \\
    0 & 0 & \ldots & 1 & \ldots & 0 & 0 \\
    0 & 0 & \ldots & 0 & \ddots & 0 & 0 \\
    0 & 0 & \ldots & 0 & \ldots & 1 & 0
\end{pmatrix}$$

and it was found that $r_{k,k-1} = 0$ or $|r_{k,k-1}|$ is very small. It is then possible to continue the transformation by the Danilevsky method.

Two cases are possible here.

**Case 1.** Suppose for some $j = 1, 2, \ldots, k-2$, $r_{kj} \neq 0$. Then by permuting the $jth$ row and $(k-1)th$ row and the $jth$ column and $(k-1)th$ column of $R$ we obtain a matrix $R' = (r'_{ij})$ similar to $R$ with $r_{k,k-1}' \neq 0$. 

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Case 2. Suppose now that \( r_{kj} = 0 \) for all \( j = 1, 2, \ldots, k - 2 \). Then \( R \) is in the form

\[
R = \begin{bmatrix}
R_1 & R_2 \\
0 & R_3
\end{bmatrix} = \begin{pmatrix}
r_{11} & r_{12} & \ldots & r_{1,k-1} & r_{1,k} & r_{1,k+1} & \ldots & r_{1n} \\
r_{21} & r_{22} & \ldots & r_{2,k-1} & r_{2,k} & r_{2,k+1} & \ldots & r_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
r_{k-1,1} & r_{k-1,2} & \ldots & r_{k-1,k-1} & r_{k-1,k} & r_{k-1,k+1} & \ldots & r_{k-1,n} \\
0 & 0 & \ldots & 0 & r_{kk} & r_{k,k+1} & \ldots & r_{kn} \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix}.
\]

In this case the characteristic polynomial of \( R \) breaks up into two determinants:

\[
\det(\lambda I_n - R) = \det(\lambda I_{k-1} - R_1) \det(\lambda I_{n-k+1} - R_3).
\]

Here, the matrix \( R_3 \) is already reduced to the Frobenius form. It remains to apply the Danilevsky’s method to the matrix \( R_1 \).

Note. Since \( U_k A_{k-1} \) only changes the \( k \)th row of \( A_{k-1} \), it is more efficient to multiply first \( A_{k-1} \) by its \((k+1)\)th row and then multiply on the right side the resulting matrix by \( V_k \).

The next result shows that once we transform \( A \) into its Frobenius form; we may obtain the eigenvectors with the help of the matrices \( V_i' \)’s.

**Theorem.** Let \( A \) be an \( n \times n \) matrix and let \( F[A] \) be its Frobenius form. If \( \lambda \) is an eigenvalue of \( A \), then

\[
v = \begin{pmatrix}
\lambda^{n-1} \\
\lambda^{n-2} \\
\vdots \\
\lambda \\
1
\end{pmatrix}
\]

and

\[
vw = V_{n-1}V_{n-2} \cdots V_2 V_1 v
\]

are the eigenvectors of \( F[A] \) and \( A \) respectively.

**Proof.** Since

\[
\det(\lambda I_n - A) = \det(\lambda I_n - F[A]) = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1} \lambda + p_n,
\]

we have

\[
(\lambda I_n - F[A])v = \begin{pmatrix}
\lambda - p_1 & -p_2 & \ldots & -p_{n-1} & -p_n \\
1 & \lambda & 0 & \ldots & 0 \\
\lambda & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix} \begin{pmatrix}
\lambda^{n-1} \\
\lambda^{n-2} \\
\vdots \\
\lambda \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Since \( F[A] = V_{n-1}^{-1}V_{n-2}^{-1} \cdots V_2^{-1} V_1^{-1} AV_{n-1}V_{n-2} \cdots V_2 V_1 \) and \( F[A]v = \lambda v \), we conclude that

\[
\lambda w = V_{n-2} \cdots V_2 V_1 (\lambda v) = (V_{n-2} \cdots V_2 V_1) F[A]v = A(V_{n-1}V_{n-2} \cdots V_2 V_1) v = Aw
\]

**Note.** For expanding characteristic polynomials of matrices of order higher than fifth, the method of Danilevsky requires less multiplications and additions than other methods.

**Example.** Reduce the matrix

\[
A = \begin{pmatrix}
1 & 1 & 3 & 4 \\
2 & 0 & 2 & 1 \\
1 & 0 & 1 & 2 \\
0 & -1 & -1 & -1
\end{pmatrix}
\]

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to its Frobenius form.

The matrix $B = A_3 = U_3AV_3$ is as follows:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 1 \\ 2 & 0 & -2 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $b_{32} = 0$, we need the permutation matrix $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; thus

$$C = JBJ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 2 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 & -1 \\ 1 & 1 & -3 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Next we obtain the matrix $D = A_2 = U_2CV_2$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & -2 & -1 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 & -3 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Finally the Frobenius form $F[A] = A_1 = U_1DV_1$,

$$F[A] = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & -2 & -3 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus the Characteristic polynomial of $A$ is:

$$K_A(\lambda) = x^4 - x^3 - 4x^2 - 2x - 3$$

Matlab Program

```matlab
A = input('Enter the square matrix A : ');
m = size(A); N = m(1); b = [1]; B = zeros(N); i = 1;
while i < N,
    J = eye(N); h = A(N - i + 1, N - i)
    while h == 0;
        c = A(N - i + 1 : N - i); z = norm(c, inf);
        if z == 0;
            k = 1; r = 0;
        end while r == 0 & k < N - i;
            r = r + c(N - i - k);
    J(N - i, N - i) = 0; J(N - i, N - i - k + 1) = 1;
    J(N - i - k + 1, N - i - k + 1) = 1;
    J(N - i - k + 1, N - i) = 1;
    A = J * A * J; k = k + 1;
```
The polynomial \( \phi \) such that
$$K(x) = x \land n + c(1)x \land (n - 1) + \ldots + c(n - 1)x + c(n)$$
divides \( K x \), hence of degree strictly less than \( r \).

The polynomial \( \phi(\lambda) \) is said to annihilate \( v \) and to be minimal for \( v \). If \( \omega(\lambda) \) is another monic polynomial which annihilates \( v \),
\[
\omega(A)v = \gamma(\lambda)\phi(\lambda) + \rho(\lambda),
\]
then \( \phi(\lambda) \) divides \( \omega(\lambda) \). To show that; suppose
\[
\omega(\lambda) = \gamma(\lambda)\phi(\lambda) + \rho(\lambda),
\]
where \( \rho(\lambda) \) is the remainder after dividing \( \omega \) by \( \phi \), hence of degree strictly less than \( r \), it follows that
\[
\rho(A)v = 0.
\]
But \( \phi(\lambda) \) is minimal for \( v \), hence \( \rho(\lambda) = 0 \).

Now of all vectors \( v \) there is at least one vector for which the degree \( v \) is maximal, since for any vector \( v \), \( r(v) \leq n \). We call such vector a maximal vector.

A monic polynomial \( \mu_A(\lambda) \) is said to be the minimal polynomial for \( A \), if \( \mu_A(\lambda) \) is monic and of minimum degree satisfying

\[
\phi(A) = Z_n.
\]

**Theorem 1.** Let \( A \) be an \( n \times n \) matrix and let \( \phi(\lambda) \) be a minimal polynomial for a maximal vector \( v \). Then \( \phi(\lambda) \) is the minimal polynomial for \( A \).

**Proof.** Consider any vector \( u \) such that \( u \) and \( v \) are linearly independent. Let \( \psi(\lambda) \) be its minimal polynomial. If \( \omega \) is the lowest common multiple of \( \phi \) and \( \psi \), then \( \omega \) annihilates every vector in the plane of \( u \) and \( v \), since

\[
\omega(A)(\alpha u + \beta v) = \alpha \omega(A) u + \beta \omega(A) v = \theta.
\]

Hence \( \omega \) contains as a divisor the minimal polynomial of every vector in the plane. But \( \omega \) is of degree \( 2n \) at most, hence has only finitely many divisors. Since there are infinitely many pairs of linearly independent vectors in the plane and finitely many divisors of \( \omega \), there is a pair of linearly independent vectors \( x \) and \( y \) in this plane with the same minimal polynomial. This polynomial also annihilates \( v \) since \( v \) is on this plane. Therefore \( \phi \) is minimal for every vectors in the plane of \( u \) and \( v \), and since \( u \) was any vector whatever, other than \( v \), \( \phi \) annihilates every \( n \)-dimensional vector.

Since \( \phi \) annihilates every vector, it annihilates in particular every vector \( e_i \), hence

\[
\phi(A)I = \phi(A) = Z_n.
\]

Thus \( \phi(\lambda) = \mu_A(\lambda) \) is the minimal polynomial for \( A \).

If the minimal polynomial and the characteristic polynomial of a matrix are equal, then they may be found by the use of Krylov’s sequence. To produce the characteristic polynomial of \( A \) by Krylov method, first choose an arbitrary \( n \)-dimensional nonzero column vector \( v \) such as \( e_1 \), then use the Krylov sequence to define the matrix

\[
V = [v, Av, A^2v, \ldots, A^{n-2}v, A^{n-1}v] = [v_0, v_1, v-2, \ldots, v_{n-2}, v_{n-1}].
\]

If the matrix \( V \) has rank \( n \), then the system \( Vc = -v_n \) has a unique solution

\[
c^t = (c_0, c_1, c_2, \ldots, c_{n-1}).
\]

The monic polynomial

\[
\phi(\lambda) = c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots + c_{n-1} \lambda^{n-1} + \lambda^n
\]

which annihilates \( v \) is the characteristic polynomial of \( A \). If the system \( Vc = -v_n \) does not have a unique solution, then change the initial vector and try for example with \( e_2 \).

**Examples.** Compute the characteristic polynomials of the following matrices:

\[
A = \begin{pmatrix}
1 & 2 & 1 & -1 \\
1 & 0 & 2 & 1 \\
2 & 1 & -1 & 3 \\
4 & -5 & 0 & 4
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
1 & 2 & -3 & 1 \\
1 & 0 & -2 & 1 \\
1 & -3 & -1 & 3 \\
1 & 0 & 1 & -2
\end{pmatrix}.
\]

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Choosing the initial vector \( v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) for both matrices, we obtain

\[
V_A = [v, Av, A^2v, A^3v] = \begin{pmatrix} 1 & 1 & 1 & 17 \\ 0 & 1 & 9 & 42 \\ 0 & 2 & 13 & 43 \\ 0 & 4 & 15 & 19 \end{pmatrix}
\]

and

\[
V_B = [v, Bv, B^2v, B^3v] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

The matrix \( V_A \) is nonsingular, hence

\[
c_A = -V^{-1}_A A v = \begin{pmatrix} -87 \\ 28 \\ 2 \\ -4 \end{pmatrix}.
\]

From \( c_A \) we obtain the characteristic polynomial of \( A \) which is

\[
K_A(\lambda) = -87 + 28\lambda + 2\lambda^2 - 4\lambda^3 + \lambda^4.
\]

The matrix \( V_B \) is singular, so we need another initial vector such as \( v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \). The new matrix \( V_B = \begin{pmatrix} 0 & 2 & 11 & 11 \\ 1 & 0 & 8 & 0 \\ 0 & -3 & 5 & -21 \\ 0 & 0 & -1 & 18 \end{pmatrix} \) is invertible, so

\[
c_B = -V^{-1}_B A^4 v = \begin{pmatrix} 9 \\ -2 \\ -10 \\ 2 \end{pmatrix}.
\]

From the solution \( c_B \) we obtain the characteristic polynomial of \( B \) which is

\[
K_B(\lambda) = 9 - 2\lambda - 10\lambda^2 + 2\lambda^3 + \lambda^4.
\]

Remark. The minimal polynomial and the characteristic polynomial of the matrix

\[
A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

are

\[
m_A(\lambda) = \lambda^3 - 3\lambda^2 - 7\lambda \quad \text{and} \quad K_A(\lambda) = \lambda^4 - 3\lambda^3 - 7\lambda^2,
\]

respectively. Therefore by choosing any initial vector \( v \), the matrix \( V_A = [v, Av, A^2v, A^3v] \) will always be singular. This means that the Krylov sequence will never produce the characteristic polynomial \( K_A(\lambda) \).

Matlab Program

\[
A = \text{input('Enter a square matrix } A : ')};
\]

\[
m = \text{size}(A); \quad n = m(1); \quad V = \text{zeros}(n,n);
\]

\[
DL1 = \text{['Enter an initial ', int2str(n), ' - dimensional row vector } v0 = ']};
\]

\[
v0 = \text{input(DL1)};
\]

\[
z = 0; \quad k = 1;
\]

\[
\text{while } z == 0 \&\& k < 5 \quad \text{do}
\]

\[
w = v0; \quad V(:,1) = w;
\]

\[
\text{for } i = 2 : n, \quad w = A \ast w; \quad V(:,i) = w; \quad \text{end};
\]

\[
\text{if } \text{det}(V) \approx 0; \quad k = 8; \quad c = -\text{inv}(V) \ast A \ast w;
\]

\[
\text{end;}
\]

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else
    while k < 5
        v0 = input('The matrix V is singular, please enter another initial row vector v0: ');
        k = k + 1;
    end;
end;

z = det(V);

if k == 5;
    disp('Sorry, the Krylov method is not suited for this matrix. '), disp(''),
else:
    disp('The Characteristic polynomial looks like: '), disp(''),
    disp('K_A(x) = c(0) + C(1)x + c(2)x^2 + \cdots + c(n - 1)x^{n - 1} + x^n'), disp(''),
    disp('The coefficients list c(k) is: '), disp(''),
    disp([c', 1])
end;