Throughout history, different civilizations have had different ways of representing numbers. Some of these systems seem strange or complicated from our perspective.

The Egyptians of 3000 BC had an interesting way to represent fractions. Although they had a notation for $1/n$ (reciprocals or unit fractions) such as $1/2$ and $1/3$ and $1/4$ and so on, their notation did not allow them to write $2/5$, $3/4$ or $4/7$ as we would today. So those fractions had to be represented as a sum of such unit fractions and, furthermore, all the unit fractions were different. For example,

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4} \quad \text{and} \quad \frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{32}.$$  

The famous Rhind papyrus, dated to around 1650 BC contains a table of representations of as Egyptian fractions for odd between 5 and 101.

**Formal Definition.** An Egyptian Fraction is a sum of positive distinct unit fractions.

♣ **Practical Use of Egyptian Fraction.** If one wants to divide 5 pizzas equally among 8 diners, the Egyptian Fractions $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$ means that each diner gets half a pizza plus another eighth of a pizza, e.g. by splitting 4 pizzas into 8 halves, and the remaining pizza into 8 eighths.

Similarly, although one could divide 13 pizzas among 12 diners by giving each diner one pizza and splitting the remaining pizza into 12 parts (perhaps destroying it); note that

$$\frac{13}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$  

This means that split 6 pizzas into halves, 4 into thirds and the remaining 3 into quarters, and then give each diner one half, one third and one quarter.

Electronic circuit design is one area where Egyptian Fractions have practical use. When electrical resistors are added to a parallel circuit, the reciprocals are added to find the total resistance. An example is a parallel circuit with $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $6$ Ohm (Ω) resistors.

$$\frac{2}{\Omega} = \frac{1}{\Omega} + \frac{1}{2\Omega} + \frac{1}{3\Omega} + \frac{1}{6\Omega}.$$  

Inductance (L) in Henrys (H) in parallel circuits is calculated in the same way.

$$\frac{1}{2H} = \frac{1}{4H} + \frac{1}{15H} + \frac{1}{36H} + \frac{1}{45H}.$$  

♣ **Greedy Algorithm**. The algorithm, as the name suggests, always makes the choice that seems to be the best at that moment. This means that it makes a locally-optimal choice in the hope that this choice will lead to a globally-optimal solution.
Fibonacci’s Greedy Algorithm for Egyptian Fractions. Leonardo of Pisa (more commonly known as Fibonacci), in his book Liber Abaci (the book in which he mentions the rabbit problem involving the Fibonacci Numbers) written in 1202, showed that any rational number may be expressed as a sum of unit fractions.

The algorithm is used to find the sum of unit fractions for any rational number between zero and one.

First find $c$, the ceiling of the fraction $\frac{a}{b}$. Then find $\frac{a_1}{b_1} = \frac{a}{b} - \frac{1}{c} < \frac{a}{b}$. If $a_1 = 1$, then we stop, if not, we continue splitting the fraction into smaller fractions using the same technique, until we end up with a unit fraction.

We explain this algorithm with an example.

The fraction $\frac{521}{1050}$ is less than one-half, but it is bigger than one-third. So the largest unit fraction we can take away from $\frac{521}{1050}$ is $\frac{1}{3}$, thus

$$\frac{521}{1050} - \frac{1}{3} = \frac{171}{1050} = \frac{57}{350}.$$

Now we repeat the process on $\frac{57}{350}$ This time $\lceil \frac{57}{350} \rceil = 1$ and the remainder is $\frac{1}{50}$. So

$$\frac{521}{1050} = \frac{1}{3} + \frac{1}{7} + \frac{1}{50}.$$

Notice that each time we use this process, the fraction becomes smaller than the previous one. This means that eventually we obtain a unit fraction.

To state our main result, we need the following lemma

Lemma 1. For every positive integer $n$:

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

Proof. By taking great common divisor of the right side of the equal sign, we obtain the left side.

Theorem 1. Any rational number $\frac{a}{b}$ between 0 and 1, written in lowest term, can be represented as a sum of different unit fractions. Moreover there are infinitely many Egyptian Fractions for the same positive rational number less than one.

Proof. Since the number $\frac{a}{b}$ is less than one, by using the Greedy algorithm, we get

$$\frac{a_1}{b_1} = \frac{a}{b} - \frac{1}{c_1} < \frac{a}{b}, \quad \text{where} \quad c_1 = \left\lceil \frac{a}{b} \right\rceil.$$

If $a_1 = 1$, then we are done; else, we use $c_2 = \left\lceil \frac{a_1}{b_1} \right\rceil$ to obtain $\frac{a_2}{b_2} < \frac{a_1}{b_1}$. Eventually for some $n$, we end up with a unit fraction. So we have

$$\frac{a}{b} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \cdots + \frac{1}{c_n}.$$
This clearly completes the first part of the proof.

By using Lemma 1, we change \( \frac{1}{c_n} \) into \( \frac{1}{c_n + 1} + \frac{1}{c_n(c_n + 1)} \); which gives us one more unit fraction for the same rational number. We can repeat this process as many times as we wish. \( \square \)

No algorithm is known for producing unit fraction representations having either a minimum number of terms or smallest possible denominator. However, there are several algorithms to find sum of unit fraction, for a given rational number between zero and one.

The following result is useful to obtain Egyptian Fractions for some fractions without using the Greedy algorithm.

**Lemma 2.** Let \( M = \frac{a}{b} - 1 \) and \( c = a + b \), where \( a \) and \( b \) are two different positive integers. Then

1. \( \frac{a}{M} = \frac{1}{b} + \frac{1}{bM} \).
2. \( \frac{M}{c} = \frac{a}{M} + \frac{b}{M} = \frac{1}{b} + \frac{1}{bM} + \frac{1}{a} + \frac{1}{aM} \).
3. If \( 0 < a < b \) and \( a^{-1} \) is the inverse of \( a \) modulo \( b \); i.e. \( aa^{-1} = bq + 1 \), for some positive integer \( q \). Then \( \frac{a}{b} = \frac{1}{a^{-1}b} + \frac{q}{a^{-1}} \).

**Proof:** (1) By using the fact that \( M + 1 = \frac{ab}{b} \), we obtain

\[
\frac{1}{b} + \frac{1}{bM} = \frac{M + 1}{bM} = \frac{ab}{bM} = \frac{a}{M}.
\]

(2) The identity follows from the first part and using the fact that \( c = a + b \).

(3) The identity follows from taking the common denominator of the right side of the equal sign and using the identity \( aa^{-1} = bq + 1 \). \( \square \)

**Note.** Fibonacci applied the algebraic identity in part (2) of the above theorem to show that:

\[
\frac{8}{11} = \frac{6}{11} + \frac{2}{11} = \frac{1}{2} + \frac{1}{22} + \frac{1}{6} + \frac{1}{66}.
\]

**Note.** There are several other methods to obtain Egyptian Fractions. We present another method.

\( \blacklozenge \) **Golomb’s Method.** Consider the rational number \( \frac{a}{b} \) between 0 and 1, written in lowest term. Then by Lemma 2, part (3), we have

\[
\frac{a}{b} = \frac{1}{a^{-1}b} + \frac{q}{a^{-1}}.
\]

If \( q = 1 \), then we have \( \frac{a}{b} \) as the sum of two unit fractions; otherwise we can apply the above procedure for \( \frac{q}{a^{-1}} \). We keep repeating until we end up with only unit fractions.

**Example.** Consider the fraction \( \frac{8}{11} \).

**Step 1.** \( 8(7) = 11(5) + 1 \), so \( \frac{8}{11} = \frac{1}{77} + \frac{5}{7} \).
Step 2. \( \frac{5}{7} = \frac{1}{21} + \frac{2}{3} \).

Step 3. \( \frac{2}{3} = \frac{1}{2} + \frac{1}{6} \).

Thus

\[
\frac{8}{11} = \frac{1}{77} + \frac{1}{21} + \frac{1}{2} + \frac{1}{6}.
\]

♠ Shortest Egyptian Fractions.

• There are two ways to write \( \frac{4}{5} \) as a sum of three unit fractions:

\[
\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20} \quad \text{and} \quad \frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}.
\]

• There are many other fractions whose shortest Egyptian Fraction has 3 unit fractions. Those with a denominator 10 or less are:

\[
\frac{3}{7}, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}.
\]

• We already showed that \( \frac{8}{11} \) needs 4 unit fractions. There are 15 other Egyptian Fractions of length 4 for this fraction. Here is one of them:

\[
\frac{8}{11} = \frac{1}{2} + \frac{1}{6} + \frac{1}{22} + \frac{1}{66}.
\]

• \( \frac{16}{17} \) is the fraction with the smallest numerator and denominator:

\[
\frac{16}{17} = \frac{1}{2} + \frac{1}{3} + \frac{1}{17} + \frac{1}{34} + \frac{1}{51}.
\]

There are 38 other Egyptian Fractions of length 5 for this fraction.

• The smallest fraction needing 6 unit fractions is \( \frac{77}{79} \):

\[
\frac{77}{79} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{79} + \frac{1}{474} + \frac{1}{632}.
\]

• The smallest fraction needing 7 unit fractions is \( \frac{732}{733} \):

\[
\frac{732}{733} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{45} + \frac{1}{7330} + \frac{1}{20524} + \frac{1}{26388}.
\]

• The smallest fraction needing 8 unit fractions is \( \frac{27538}{27539} \).

Mr. Huang Zhibin of China in April 2014 has verified that this fraction needs 8 unit fractions and gives this example:

\[
\frac{27538}{27539} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1933} + \frac{1}{14893663} + \frac{1}{1927145066572824} + \frac{1}{212829231672162931784}.
\]

Beyond 8 unit fractions is an unknown territory.
Series Representing the Sum of Unit Fractions. There are several important series using the sum of unit fractions. Here are some examples:

Harmonic Series.

\[
\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.
\]

The above divergent infinite series is called Harmonic because if we stretch a string tightly and twang it, we hear a certain note. Stopping the string at the half way point makes a sound an octave above the first note. If instead we take just one third of the length we get another note that seems harmonious to the ear in relation to the whole string. We also get harmonious sounds if we take one quarter and one fifth and so on for some time. Pythagoras first noted this connection with harmonious sound and the lengths of plucked strings.

The fact that the harmonic series diverges was first proven in the 14th century by Nicole Oresme, but this achievement fell into obscurity. Proofs were given in the 17th century by Pietro Mengoli, Johann Bernoulli, and Jacob Bernoulli.

Leonhard Euler proved that the sum which includes only the reciprocals of primes also diverges, i.e.

\[
\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \cdots = \infty.
\]

It is not difficult to show that the sum of the reciprocals of all the composite numbers

\[
S = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{15} + \cdots
\]

diverges. Note that:

\[
S > \sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \cdots = \frac{1}{2} \left( \sum_{n=2}^{\infty} \frac{1}{n} \right) = \infty.
\]

The finite partial sums of the diverging harmonic series,

\[
H_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n},
\]

are called harmonic numbers.

The general harmonic series for rational numbers is of the form \(\sum_{n=0}^{\infty} \frac{1}{an+b}\), where \(a\) and \(b\) are positive integers. By the limit comparison test with the harmonic series, all general harmonic series also diverge.

P-Series. A generalization of the harmonic series is the p-series (or hyperharmonic series), defined as

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots.
\]

for any positive real number \(p\). When \(p = 1\), the p-series is the harmonic series, which diverges. The p-series converges for all \(p > 1\) and diverges for all \(p \leq 1\).
Geometric Series. The terms of a geometric series form a geometric progression, meaning that the ratio of successive terms in the series is constant. This relationship allows for the representation of a geometric series using only two terms, r and a. The term r is the common ratio, and a is the first term of the series. Here are some examples:

\[ S_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots; \quad S_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots; \quad S_5 = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots \]

The finite partial sums of any of the above series may produce a rational number which may be either less or greater than one.

See:

Egyptian Fraction