Inequalities Concerning Eigenvalues

In all that follows, we will denote by θ, the zero column vector and the identity matrix by I_n. Let $A = (aij)$ be an $n \times n$ real or complex matrix; the set of eigenvalues of A

$$\sigma(A) = \lambda_1,\ \lambda_1,\ \cdots, \lambda_1\ .$$

is called the spectrum of A. An eigenvalue with the largest modulus is called a maximal eigenvalue. The spectral radius of A denoted by $\rho(A)$ is the modulus of a maximal eigenvalue. A matrix norm is defined as:

$$||A|| = \{\max ||Av||; ||v|| = 1\}.$$

For $i = 1, 2, \dots, n$, define:

$$R_i(A) = \sum_{i=1}^{n} |a_{ij}| \quad \text{and} \quad r_i(A) = R_i(A) - |a_{ii}|.$$

The row norm of A is defined as follows:

$$||A||_\infty = \max_{1\leq i\leq n} \left\{ \sum_{j=1}^{n} a_{ij} \right\} = \max_{1\leq i\leq n} R_i(A).$$

An orthogonal matrix Q (respectively unitary matrix U) is a matrix satisfying $Q^tQ = I_n$ (respectively $U^*U = I_n$).

We need the following two lemmas:

Lemma.

$$\det(A) = \prod_{i=1}^{n} \lambda_i \quad \textit{and} \quad \textit{trace}(A) = \sum_{i=1}^{n} \lambda_i.$$

Note that similar matrices have the same determinant and trace.

Schur's lemma. *Any $n \times n$ matrixA is unitarily similar to an upper triangular matrix $T = (t_{ij})$, i.e., $T = U^*AU$ for some unitary matrix U.*

Now we can present some important inequalities concerning eigenvalues.

Theorem. *For any sub-multiplicative matrix norm $||\,.\,||$, we have $\rho(A) < ||A||$.*

Proof. Suppose $Au = \lambda\, u$, where u is a unit vector (i.e., $||u|| = 1$). Then

$$|\lambda| = |\lambda|\, ||u|| = ||\lambda\, u|| = ||Au|| \leq \{\max\ ||Av|| :\ ||v|| = 1\} = ||A||.$$

Thus $\rho(A) \leq ||A||$. ■

Levy-Deslanque theorem. *If the matrix A is strictly diagonally dominant, that is*

$$|a_{ii}| > r_i(A) \quad \textit{for all} \quad i = 1, 2, \dots, n.$$

Then A is invertible.

Proof. Suppose $\det(A) = 0$, then for some nonzero vector $u = (u_1, u_2, \dots, u_n)^t$, $A\,u = \theta$. Now let k be the index where

$$u_k \geq u_i \qquad \text{for all} \qquad i = 1, 2, \dots, n\,.$$

Then

$$|a_{kk}|\ ||uk|| = \left|\sum_{i \neq j} a_{kj}\, u_j\ \right| \leq \sum_{i \neq j} |a_{kj}|\, u_j| \leq r_i(A)\cdot$$

which is a contradiction with $|akk| > r_k(A)$. ■

A generalization of Levy-Deslanque theorem is presented without a proof.

Ovals of Cassini. *If*

$$|aii|\,|ajj| > r_i(A)_j(A) \quad (i = 1, 2, ..., n \quad \textit{and} \quad i \neq j),$$

then A is invertible.

Gerschgorin's Disks theorem. *The eigenvalues of A lie in the union of the disks*

$$D_i(a_{ii}, r_i(A)), \quad \textit{centered at} \;\; a_{ii} \quad \textit{with the radius} \quad r_i(A).$$

Moreover, the union of any k of these disks that do not intersect the remaining $(n-k)$ contains precisely k (counting multiplicities) of the eigenvalues.

Proof. We only prove the first part.
Let k be an eigenvalue of A, then $\det(A - \lambda_k\, I_n) = 0$. By the Levy-Deslanque theorem, we conclude that $|\lambda_k - a_{ii}| < r_i(A)$ for at least one i. ■

The fact that $\sigma(A) = \sigma(A^t)$, we can obtain similar results by using columns of A instead of its rows.

Example . Consider the symmetric matrix

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & -1 & 9 \end{pmatrix}$$

The eigenvalues are real and according to the above theorem, there are two eigenvalues between -1 and 5 and the dominant eigenvalue is in $[6, 12]$.

The Frobenius norm of an $m \times n$ matrix A defined as the square root of the sum of the absolute squares of its elements; it is also equal to the square root of the trace of the positive semi-definite matrix AA^*

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{trace(AA^*)} \ . \tag{6}$$

Schur's Inequalities.

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2 = ||A||_F^2 \cdot$$

Proof. According to Schur's lemma, $T = U^*AU$ for some upper triangular matrix $T = (t_{ij})$ and unitary matrix $U = (u_{ij})$. Thus

$$T^* = U^*A^*U \quad \text{and} \quad TT^* = (U^*AU)(U^*A^*U) = U^*AA^*U \cdot$$

The facts that $trace(AA^*) = trace(TT^*)$ and $trace(AA^*) = \sum_{i,j=1}^{n} |a_{ij}|^2$ imply that

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = trace(AA^*) = trac(TT^*) = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i=1}^{n} |t_{ii}|^2 \cdot$$

Hence the desired conclusion. ■