Jordan normal form or Jordan canonical form (named in honor of Camille Jordan) shows that by changing the basis, a given square matrix $M$ can be transformed into a certain normal form

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & J_p \end{pmatrix},$$

where each block $J_i$ is a square matrix of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}.$$

Note that $\lambda_i's$ are the repeated eigenvalues of $M$. 1. Given an eigenvalue $\lambda_i$ of an $n \times n$ matrix $M$, its geometric multiplicity is the dimension of $\text{Ker}(M - \lambda_i I_n)$, and it is the number of Jordan blocks corresponding to $\lambda_i$.

2. The sum of the sizes of all Jordan blocks corresponding to an eigenvalue $\lambda_i$ is its algebraic multiplicity.

3. $M$ is diagonalizable if and only if, for any eigenvalue $\lambda$ of $M$, its geometric and algebraic multiplicities coincide.

A non-diagonalizable matrix is sometimes called a defective matrix.

To obtain a Jordan normal form of an $n \times n$ diagonalizable matrix $M$, we use $n$ linearly independent eigenvectors to construct a matrix $P$, where the diagonal matrix

$$P^{-1}MP = J(M)$$

will be the Jordan normal form.

If $\lambda$ is an eigenvalue of a defective $n \times n$ matrix $A$ with an algebraic multiplicity greater than one and a geometric multiplicity less than its algebraic multiplicity, then any nonzero vector $v$ satisfying:

$$[M - \lambda I_n]^k v = \theta \quad \text{for } k = 2, 3, \ldots$$

is called a generalized eigenvector.

To construct the Jordan canonical form of $M$, we form a sequence of generalized eigenvectors that satisfy:

$$[M - \lambda I_n] v_1 = u, \quad [M - \lambda I_n] v_2 = v_1, \quad [M - \lambda I_n] v_3 = v_2, \quad \cdots \quad \cdots$$

The eigenvectors and generalized eigenvectors of $M$ form the columns of the invertible matrix $P$ which gives us the Jordan normal form $J(M)$. 

Suppose the geometric multiplicity of \( \lambda \) is \( m \) and its algebraic multiplicity is larger than \( m \). First, we need to find
\[
M \mathbf{u}_1 = \lambda \mathbf{u}_1, \quad M \mathbf{u}_2 = \lambda \mathbf{u}_2, \quad \ldots , \quad M \mathbf{u}_m = \lambda \mathbf{u}_m;
\]
then we obtain \( v_{jk}'s \), the generalized eigenvectors produced by \( u_j's \). The matrix \( P \) will be formed as follows:
\[
P = [u_1 \ v_{11} \ v_{12} \ \ldots \ v_{1r} \ u_2 \ v_{21} \ v_{22} \ \ldots \ v_{2s} \ u_3 \ \ldots \ldots].
\]
If we are unable to produce generalized eigenvectors with \( u_j's \), then we need to find other linearly independent eigenvectors.

\section*{Repeated Eigenvalues}

Consider the following \( 4 \times 4 \) matrices:
\[
A = \begin{pmatrix}
-30 & -12 & 19 & 27 \\
-10 & 1 & 5 & 8 \\
-6 & -2 & 7 & 5 \\
-43 & -15 & 24 & 38
\end{pmatrix}, \quad B = \begin{pmatrix}
-12 & -6 & 9 & 13 \\
-1 & 4 & 0 & 1 \\
-6 & -2 & 7 & 5 \\
-16 & -6 & 9 & 17
\end{pmatrix},
\]
\[
C = \begin{pmatrix}
-25 & -10 & 16 & 23 \\
-20 & -3 & 11 & 16 \\
-11 & -4 & 10 & 9 \\
-38 & -13 & 21 & 34
\end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix}
-7 & -4 & 6 & 9 \\
-11 & 0 & 6 & 9 \\
-11 & -4 & 10 & 9 \\
-11 & -4 & 6 & 13
\end{pmatrix},
\]
with the characteristic polynomials:
\[
K_A(\lambda) = K_B(\lambda) = K_C(\lambda) = K_D(\lambda) = \lambda^4 - 16\lambda^3 + 96\lambda^2 - 256\lambda + 256 = (\lambda - 4)^4.
\]
Thus \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 4 \).

\textbf{Case 1.} The geometric multiplicity of the matrix \( A \) is one, so there is only one Jordan block.

The rank of the matrix \( \hat{A} = A - 4I_4 \) is 3, so the vector \( u(A) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) which
\[
is the solution of the homogeneous system \( \hat{A}u = \theta \) is the only linearly independent eigenvector of \( A \).

The vectors
\[
v(A) = \begin{pmatrix}
-1 \\
-2 \\
-3 \\
0
\end{pmatrix}, \quad w(A) = \begin{pmatrix}
3 \\
9 \\
11 \\
0
\end{pmatrix}, \quad \text{and} \quad z(A) = \begin{pmatrix}
0 \\
-16 \\
-27 \\
12
\end{pmatrix}
\]
are the generalized eigenvectors of \( A \), obtain by solving
\[
\hat{A}v = u(A), \quad \hat{A}w = v(A), \quad \text{and} \quad \hat{A}z = w(A) \text{ respectively.}
\]

By constructing the matrix
\[
P = [u(A) \ v(A) \ w(A) \ z(A)] = \begin{pmatrix}
1 & -1 & 3 & 0 \\
1 & -2 & 9 & -16 \\
1 & -3 & 11 & -27 \\
1 & 0 & 0 & 12
\end{pmatrix}
\]
\[
P^{-1} = \begin{pmatrix}
-60 & -24 & 36 & 49 \\
-43 & -15 & 24 & 34 \\
6 & 3 & -4 & -5 \\
5 & 2 & -3 & -4
\end{pmatrix}, \quad \text{with}
\]
we obtain the Jordan canonical form of $A$ as follows:

$$J(A) = P^{-1}AP = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

**Case 2a.** The geometric multiplicity of the matrix $B$ is two, so there are two blocks in the Jordan normal matrix. Since the rank of the matrix

$$\tilde{B} = B - 4I_4 = \begin{pmatrix} -16 & -6 & 9 & 13 \\ -1 & 0 & 0 & 1 \\ -6 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \end{pmatrix}$$

is 2, the vectors

$$u_1(B) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad u_2(B) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

which are the solution of the homogeneous system $\tilde{B}u = \theta$ are linearly independent eigenvectors of $B$. To obtain the Jordan normal form of $B$, we need to obtain two generalized eigenvectors of $B$ by solving the systems $\tilde{B}v = u_1(B)$ and $\tilde{B}v = u_2(B)$. The generalized eigenvectors are

$$v_1(B) = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2(B) = \begin{pmatrix} -2 \\ 0 \\ -5 \\ 1 \end{pmatrix}.$$ 

The Jordan normal form of $B$ is obtain from the matrix

$$Q = [u_1(B) \ v_1(B) \ u_2(B) \ v_2(B)] = \begin{pmatrix} 1 & -1 & 0 & -2 \\ 1 & 1 & 3 & 0 \\ 1 & -1 & 2 & -5 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

with

$$Q^{-1} = \begin{pmatrix} -5 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \\ 7 & 3 & -4 & -6 \\ 5 & 2 & -3 & -4 \end{pmatrix}.$$ 

We have

$$J(B) = Q^{-1}BQ = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$ 

**Case 2b.** The geometric multiplicity of the matrix $C$ is also two, so there are two blocks in the Jordan normal form. Since the rank of the matrix

$$\tilde{C} = C - 4I_4 = \begin{pmatrix} -29 & -10 & 16 & 23 \\ -20 & -7 & 11 & 16 \\ -11 & -4 & 6 & 9 \\ -38 & 13 & 21 & 30 \end{pmatrix}$$


is 2, the vectors
\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
3 \\
-1 \\
2
\end{pmatrix}
\]
are linearly independent eigenvectors of \( C \). By solving the system \( \hat{C}v = u_1(C) \), we obtain two linearly independent generalized eigenvectors
\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 \\
0 \\
-1 \\
2
\end{pmatrix}
\]
but the vectors \( u_1(C), u_2(C), v_{11}(C) \) and \( v_{12}(C) \) are linearly dependent; so we should use one of the two options:
1. Remove one of the generalized eigenvectors and solve the system \( \hat{C}v = u_2(C) \).
2. Solve either \( \hat{C}w = v_{11}(C) \) or \( \hat{C}w = v_{12}(C) \).
The system \( \hat{C}v = u_2(C) \) is inconsistent, so we use the second option. The system \( \hat{C}w = v_{11}(C) \) is also inconsistent, but \( \hat{C}w = v_{12}(C) \) has the vector \( w_{12}(C) = \begin{pmatrix} -2 \\ 8 \\ 0 \\ 1 \end{pmatrix} \) as a solution. Now by choosing \( u_1(C), u_2(C), v_{12}(C) \) and \( w_{12}(C) \), we may construct the matrix
\[
R = \begin{bmatrix}
u_1(C) & v_{12}(C) & w_{12}(C) & u_2(C)\end{bmatrix} = \begin{pmatrix}
1 & 1 & -2 & 0 \\
1 & 0 & 8 & 3 \\
1 & -1 & 0 & -1 \\
1 & 2 & 1 & 2
\end{pmatrix}, \quad \text{with} \quad R^{-1} = \begin{pmatrix}
3 & 1 & -1 & -2 \\
-20 & -7 & 11 & 16 \\
-9 & -3 & 5 & 7 \\
23 & 8 & -13 & -18
\end{pmatrix}.
\]
The Jordan canonical form of \( C \) is as follows:
\[
J(C) = R^{-1}CR = \begin{pmatrix}
4 & 1 & 0 \\
0 & 4 & 1 \\
0 & 0 & 4 \\
0 & 0 & 0
\end{pmatrix}.
\]

**Case 3.** The rank of the matrix \( \hat{D} = D - 4I_4 = \begin{pmatrix}
-11 & -4 & -6 & 9 \\
-11 & -4 & 6 & 9 \\
-11 & -4 & 6 & 9 \\
-11 & -4 & 6 & 9
\end{pmatrix} \) is one, so there are three blocks in the Jordan normal form. The three linearly independent eigenvectors of \( D \) are as follows:
\[
u_1(D) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2(D) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \quad \text{and} \quad u_3(D) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -2 \end{pmatrix}.
\]
The generalized eigenvector \( v_1(D) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} \) is a solution of the system \( \hat{D}v = u_1(D) \).

Note that the eigenvectors \( \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \\ 3 \end{pmatrix} \) are linearly independent but do not produce any
generalized eigenvector since all the rows of \( \hat{D} \) are identical but components of these three vectors are not all equal. The matrix

\[
S = [u_1(D) \ v_1(D) \ u_2(D) \ u_3(D)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & -1 & 0 & -2 \end{pmatrix}
\]

with \( S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -11 & -4 & 6 & 9 \\ -4 & -1 & 2 & 3 \\ 6 & 2 & -3 & -5 \end{pmatrix} \)

will produce the Jordan normal form as follows:

\[
J(B) = Q^{-1}BQ = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}
\]

**Remark 1.** The following vectors

\[
u_1'(D) = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}, \quad u_2'(D) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \quad \text{and} \quad u_3'(D) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -2 \end{pmatrix}
\]

are also linearly independent eigenvectors of the matrix \( D \); but none of them could produce a generalized eigenvector \( \nu \) which is needed to obtain \( J(D) \). Since \( \hat{D}^2 \) is the zero matrix, it follows that any vector is a generalized eigenvector of \( D \); for example the first three columns of the invertible matrix

\[
T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix}
\]

are the eigenvectors of \( D \), and the forth column is a generalized eigenvector in the generalized sense. Unfortunately

\[
T^{-1}DT = \begin{pmatrix} 4 & 0 & 0 & 9 \\ 0 & 4 & 0 & -9 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}
\]

is not the Jordan canonical form of \( D \). In this case, we need to go back to our eigenvectors and choose the ones that will produce a Jordan canonical form.

**Example.** Consider the matrix

\[
M = \begin{pmatrix} 7 & 4 & 3 & -7 \\ 3 & 5 & 2 & -5 \\ 1 & 0 & 2 & 0 \\ 5 & 4 & 3 & -5 \end{pmatrix}
\]

with the characteristic polynomial

\[
K_M(\lambda) = \lambda^4 - 9\lambda^3 + 29\lambda^2 - 39\lambda + 18.
\]

The eigenvalues are \( \lambda_1 = 1, \lambda_2 = 2, \) and \( \lambda_3 = \lambda_4 = 3. \) with the corresponding eigenvectors:

\[
u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad u_3 = u_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]
The equation \((M - 3I_4)v = u_3\) produces the generalized eigenvector \(v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\). Thus we have

\[
P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \text{ with } P^{-1} = \begin{pmatrix} -1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 2 & 1 & -3 \end{pmatrix}.
\]

Hence the Jordan canonical form \(J(M)\) of \(M\) is obtained as follows:

\[
P^{-1}M P = \begin{pmatrix} -1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} = J(M).
\]

**System of Linear Differential Equations.** Consider the system

\[
X'(t) = AX(t) \quad \text{with} \quad P^{-1}AP = J(A).
\]

Let \(X(t) = PY(t)\), then \(X'(t) = PY'(t)\), \(P^{-1}X(t) = Y(t)\), and \(P^{-1}X'(t) = Y'(t)\). We have:

\[
Y'(t) = P^{-1}X'(t) = P^{-1}AX'(t) = P^{-1}APY(t) = J(A)Y(t).
\]

Thus by solving \(Y'(t) = J(A)Y(t)\), we may obtain the solution of the original system as follows:

\[
X'(t) = PY'(t) = P[J(A)Y(t)] = P[P^{-1}APY(t)] = [PP^{-1}]APY(t) = A[PY(t)] = AX(t).
\]

Consider the following matrices.

\[
A = \begin{bmatrix} 5 & 9 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 4 & 8 \\ -1 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}.
\]

**Step 1.** The characteristic polynomial of \(A\) and \(B\) are as follows:

\[
K_A(\lambda) = \det (A - \lambda I_3) = \begin{vmatrix} 5 - \lambda & 9 & -2 \\ -1 & -1 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = (2 - \lambda) \begin{vmatrix} 5 - \lambda & 9 \\ -1 & -1 - \lambda \end{vmatrix}
\]

\[
= (2 - \lambda) \left[ (5 - \lambda)(-1 - \lambda) + 9 \right] = (2 - \lambda)^3 \quad \text{and}
\]

\[
K_B(\lambda) = \det(B - \lambda I_3) = \begin{vmatrix} 4 - \lambda & 4 & 8 \\ -1 & -\lambda & -4 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = (2 - \lambda) \begin{vmatrix} 4 - \lambda & 4 \\ -1 & -\lambda \end{vmatrix}
\]

\[
= (2 - \lambda) \left[ -\lambda(4 - \lambda) + 4 \right] = (2 - \lambda)^3.
\]

By setting \(K_A(\lambda)\) to zero, we obtain \(\lambda_1 = \lambda_2 = \lambda_3 = 2\). Hence the multiplicity of 2 in the \(K_A(\lambda)\) is 3. We obtain the same eigenvalues by setting \(K_B(\lambda)\) to zero; the multiplicity of 2 in the \(K_A(\lambda)\) is also 3.

**Step 2.** Next, we are going to see if \(A\) and \(B\) are diagonalizable. We have:

\[
A - 2I_3 = \begin{bmatrix} 3 & 9 & -2 \\ -1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B - 2I_3 = \begin{bmatrix} 2 & 4 & 8 \\ -1 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}.
\]
The nullity of $A - 2I_3$ is 1; but the nullity of $B - 2I_3$ is 2. Hence there is one eigenvector for $A$, corresponding to the eigenvalue $\lambda = 2$; but two linearly independent eigenvectors for $B$, corresponding to the eigenvalue $\lambda = 2$. Therefore neither $A$ nor $B$ are diagonalizable.

**Step 3.** To find the eigenvectors of $A$ and $B$, we need to solve the following homogeneous linear systems:

$$[A - 2I_3 \mid 0] = \begin{bmatrix} 3 & 9 & -2 & \mid & 0 \\ -1 & -3 & 1 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix} \quad \text{and} \quad [B - 2I_3 \mid 0] = \begin{bmatrix} 2 & 4 & 8 & \mid & 0 \\ -1 & -2 & -4 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix}.$$  

By using some row operations, we obtain:

$$[A - 2I_3 \mid 0] = \begin{bmatrix} 0 & 0 & 1 & \mid & 0 \\ -1 & -3 & 1 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix} \quad \text{and} \quad [B - 2I_3 \mid 0] = \begin{bmatrix} 0 & 0 & 0 & \mid & 0 \\ -1 & -2 & -4 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix}.$$

By solving the above systems, we find

$$u(A) = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \quad u_1(B) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_2(B) = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}.$$  

Notice that $u_1(B)$ and $u_2(B)$ are linearly independent.

**Step 4.** Next, we need to find generalized eigenvectors for $A$ and $B$, corresponding to the eigenvalue $\lambda = 2$. From

$$[A - 2I_3 \mid u(A)] = \begin{bmatrix} 3 & 9 & -2 & \mid & 3 \\ -1 & -3 & 1 & \mid & -1 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 & \mid & 0 \\ -1 & -3 & 1 & \mid & -1 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix},$$

we obtain the linearly independent generalized eigenvectors:

$$v_1(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_2(A) = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}.$$  

But the matrix $P = [u(A), v_1(A), v_2(A)] = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$ is singular, so we need to find a second generation generalized eigenvector. For this, we solve the following system, using $v_1(A)$ (we could also use $v_2(A)$):

$$[A - 2I_3 \mid v_1(A)] = \begin{bmatrix} 3 & 9 & -2 & \mid & 1 \\ -1 & -3 & 1 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 & \mid & 1 \\ -1 & -3 & 1 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix},$$

the vector $w_{11} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a solution and the matrix

$$Q = [u(A), v_1(A), w_{11}(A)] = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
has an inverse $Q^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. The Jordan normal form of $A$ is

$$J(A) = Q^{-1}AQ = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

To find the Jordan normal form of $B$, we only need to find one generalized eigenvector; using either $u_1(B)$ or $u_2(B)$; we just choose $u_1(B)$. From

$$[B - 2I_3 | u(A)] = \begin{bmatrix} 2 & 4 & 8 & 2 \\ -1 & -2 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we obtain the linearly independent generalized eigenvector $v_{11}(B) = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Note that $v_{12}(B) = \begin{bmatrix} 1 \\ 0 \\ 0b \end{bmatrix}$ is another generalized eigenvector, but does not produce a non-singular matrix.

The matrix

$$R = [u_1(B), v_{11}(B), u_2(B)] = \begin{bmatrix} 2 & -1 & 4 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

has an inverse $R^{-1} = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$. The Jordan normal form of $A$ is

$$J(A) = R^{-1}AR = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

**Exercise.** Consider, the following matrices:

$$A = \begin{pmatrix} 3 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

with Characteristic Polynomials:

$$K_A(\lambda) = K_B(\lambda) = K_C(\lambda) = K_D(\lambda) = (\lambda - 2)^4.$$ 

(i) Find the Jordan Canonical forms $J(A)$, $J(B)$, $J(C)$, and $J(D)$ and invertible matrices $P_A$, $P_B$, $P_C$, and $P_D$;
such that

\[ P_A^{-1} A P_A = J(A), \quad P_B^{-1} B P_B = J(B), \quad P_C^{-1} C P_C = J(C), \quad \text{and} \quad P_D^{-1} D P_D = J(D). \]

(ii) Let

\[ X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}. \]

Solve the following systems of linear differential equations:

\[ X'(t) = A X(t), \quad X'(t) = B X(t), \quad X'(t) = C X(t), \quad \text{and} \quad X'(t) = D X(t). \]

Consider the following matrices

\[
A = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 5 & -8 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 1 & 4 \\ -4 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix}.
\]

Then find the Jordan canonical form of \( A \) and \( B \).