Jordan Canonical Form

Jordan normal form or Jordan canonical form (named in honor of Camille Jordan) shows that by changing the basis, a given square matrix M can be transformed into a certain normal form

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & J_p \end{pmatrix},$$

where each block J_i is a square matrix of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}.$$

Note that $\lambda_i' s$ are the repeated eigenvalues of M.

1. Given an eigenvalue λ_i of an $n \times n$ matrix M, its *geometric multiplicity* is the dimension of $Ker(M - \lambda_i I_n)$, and it is the number of Jordan blocks corresponding to λ_i.
2. The sum of the sizes of all Jordan blocks corresponding to an eigenvalue λ_i is its algebraic multiplicity.
3. M is *diagonalizable* if and only if, for any eigenvalue λ of M, its *geometric* and *algebraic multiplicities* coincide.

A non diagonalizable matrix is sometimes called a *defective matrix*.

To obtain a Jordan normal form of an $n \times n$ diagonalizable matrix M, we use n linearly independent eigenvectors to construct a matrix P, where the diagonal matrix

$$\mathbf{P^{-1}MP = J(M)}$$

will be the Jordan normal form.

Next we shall discuss the case of defective matrices.

If λ is an eigenvalue of a defective $n \times n$ matrix A with an algebraic multiplicity greater than one and a geometric multiplicity less than its algebraic multiplicity, then any nonzero vector v satisfying:

$$[\mathbf{M} - \lambda\,\mathbf{I_n}]^{\mathbf{k}}\,\mathbf{v} = \theta \qquad \textbf{for} \quad \mathbf{k = 2, 3}, \dots$$

is called a generalized eigenvector.

To construct the Jordan canonical form of M, we form a sequence of generalized eigenvectors that satisfy:

$$[\mathbf{M} - \lambda\,\mathbf{I_n}]\,\mathbf{v_1} = \mathbf{u}, \quad [\mathbf{M} - \lambda\,\mathbf{I_n}]\,\mathbf{v_2} = \mathbf{v_1}, \quad [\mathbf{M} - \lambda\,\mathbf{I_n}]\,\mathbf{v_3} = \mathbf{v_2}, \;\cdots \quad \cdots$$

The eigenvectors and generalized eigenvectors of M form the columns of the invertible matrix P which gives us the Jordan normal form $J(M)$.

Suppose the geometric multiplicity of λ is $\mathbf{m}$ and its algebraic multiplicity is larger than $\mathbf{m}$. First, we need to find

$$\mathbf{M}\,\mathbf{u_1} = \lambda\mathbf{u_1},\; \mathbf{M}\,\mathbf{u_2} = \lambda\mathbf{u_2},\; \cdots\;\cdots,\; \mathbf{M}\,\mathbf{u_m} = \lambda\mathbf{u_m};$$

then we obtain $v_{jk}{}'s$, the generalized eigenvectors produced by $u_j{}'s$. The matrix P will be formed as follows:

$$\mathbf{P} = [\mathbf{u_1}\;\mathbf{v_{11}}\;\mathbf{v_{12}}\;\ldots\;\mathbf{v_{1r}}\;\mathbf{u_2}\;\mathbf{v_{21}}\;\mathbf{v_{22}}\;\ldots\;\mathbf{v_{2s}}\;\mathbf{u_3}\;\ldots\;\ldots].$$

If we are unable to produce generalized eigenvectors with $u_j{}'s$, then we need to find other linearly independent eigenvectors.

♣ **Repeated Eigenvalues**. Consider the following 4×4 matrices:

$$A = \begin{pmatrix} -30 & -12 & 19 & 27 \\ -10 & 1 & 5 & 8 \\ -6 & -2 & 7 & 5 \\ -43 & -15 & 24 & 38 \end{pmatrix}, \qquad B = \begin{pmatrix} -12 & -6 & 9 & 13 \\ -1 & 4 & 0 & 1 \\ -6 & -2 & 7 & 5 \\ -16 & -6 & 9 & 17 \end{pmatrix},$$

$$C = \begin{pmatrix} -25 & -10 & 16 & 23 \\ -20 & -3 & 11 & 16 \\ -11 & -4 & 10 & 9 \\ -38 & -13 & 21 & 34 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} -7 & -4 & 6 & 9 \\ -11 & 0 & 6 & 9 \\ -11 & -4 & 10 & 9 \\ -11 & -4 & 6 & 13 \end{pmatrix}$$

with the characteristic polynomials:

$$\mathbf{K_A}(\lambda) = \mathbf{K_B}(\lambda) = \mathbf{K_C}(\lambda) = \mathbf{K_D}(\lambda) = \lambda^{\mathbf{4}} - \mathbf{16}\lambda^{\mathbf{3}} + \mathbf{96}\lambda^{\mathbf{2}} - \mathbf{256}\lambda + \mathbf{256} = (\lambda - \mathbf{4})^{\mathbf{4}}.$$

Thus $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 4$.

Case 1. The geometric multiplicity of the matrix A is one, so there is only one Jordan block.

The rank of the matrix $\widehat{A} = A - 4I_4 = \begin{pmatrix} -34 & -12 & 19 & 27 \\ -10 & -3 & 5 & 8 \\ -6 & -2 & 3 & 5 \\ -43 & -15 & 24 & 34 \end{pmatrix}$ is 3, so the vector $u(A) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ which is the solution of the homogeneous system $\widehat{A}u = \theta$ is the only linearly independent eigenvector of A.

The vectors

$$v(A) = \begin{pmatrix} -1 \\ -2 \\ -3 \\ 0 \end{pmatrix}, \quad w(A) = \begin{pmatrix} 3 \\ 9 \\ 11 \\ 0 \end{pmatrix}, \quad \text{and} \quad z(A) = \begin{pmatrix} 0 \\ -16 \\ -27 \\ 12 \end{pmatrix}$$

are the generalized eigenvectors of A, obtain by solving

$$\widehat{A}v = u(A), \quad \widehat{A}w = v(A), \quad \text{and} \quad \widehat{A}z = w(A) \text{ respectively.}$$

By constructing the matrix

$$P = [u(A)\ v(A)\ w(A)\ z(A)] = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 1 & -2 & 9 & -16 \\ 1 & -3 & 11 & -27 \\ 1 & 0 & 0 & 12 \end{pmatrix}$$

with

$$P^{-1} = \begin{pmatrix} -60 & -24 & 36 & 49 \\ -43 & -15 & 24 & 34 \\ 6 & 3 & -4 & -5 \\ 5 & 2 & -3 & -4 \end{pmatrix},$$

we obtain the Jordan canonical form of A as follows:

$$J(A) = P^{-1}AP = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Case 2a. The geometric multiplicity of the matrix B is two, so there are two blocks in the Jordan normal matrix. Since the rank of the matrix

$$\widehat{B} = B - 4I_4 = \begin{pmatrix} -16 & -6 & 9 & 13 \\ -1 & 0 & 0 & 1 \\ -6 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \end{pmatrix}$$

is 2, the vectors

$$u_1(B) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad u_2(B) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

which are the solution of the homogeneous system $\widehat{B}u = \theta$ are linearly independent eigenvectors of B. To obtain the Jordan normal form of B, we need to obtain two generalized eigenvectors of B by solving the systems $\widehat{B}v = u_1(B)$ and $\widehat{B}v = u_2(B)$. The generalized eigenvectors are

$$v_1(B) = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2(B) = \begin{pmatrix} -2 \\ 0 \\ -5 \\ 1 \end{pmatrix}.$$

The Jordan normal form of B is obtain from the matrix

$$Q = [u_1(B)\ v_1(B)\ u_2(B)\ v_2(B)] = \begin{pmatrix} 1 & -1 & 0 & -2 \\ 1 & 1 & 3 & 0 \\ 1 & -1 & 2 & -5 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

with

$$Q^{-1} = \begin{pmatrix} -5 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \\ 7 & 3 & -4 & -6 \\ 5 & 2 & -3 & -4 \end{pmatrix}.$$

We have

$$J(B) = Q^{-1}BQ = \left(\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right).$$

Case 2b. The geometric multiplicity of the matrix C is also two, so there are two blocks in the Jordan normal form. Since the rank of the matrix

$$\widehat{C} = C - 4I_4 = \begin{pmatrix} -29 & -10 & 16 & 23 \\ -20 & -7 & 11 & 16 \\ -11 & -4 & 6 & 9 \\ -38- & 13 & 21 & 30 \end{pmatrix}$$

is 2, the vectors

$$u_1(C) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad u_2(C) = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 2 \end{pmatrix}$$

are linearly independent eigenvectors of C. By solving the system $\widehat{C}v = u_1(C)$, we obtain two linearly independent generalized eigenvectors

$$v_{11}(C) = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v_{12}(B) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

but the vectors $u_1(C)$, $u_2(C)$, $v_{11}(C)$ and $v_{12}(C)$ are linearly dependent; so we should use one of the two options:

1. Remove one of the generalized eigenvectors and solve the system $\widehat{C}v = u_2(C)$.
2. Solve either $\widehat{C}w = v_{11}(C)$ or $\widehat{C}w = v_{12}(C)$.

The system $\widehat{C}v = u_2(C)$ is inconsistent, so we use the second option. The system $\widehat{C}w = v_{11}(C)$ is also inconsistent, but $\widehat{C}w = v_{12}(C)$ has the vector $w_{12}(C) = \begin{pmatrix} -2 \\ 8 \\ 0 \\ 1 \end{pmatrix}$ as a solution. Now by choosing $u_1(C)$, $u_2(C)$, $v_{12}(C)$ and $w_{12}(C)$, we may construct the matrix

$$R = [u_1(C)\ v_{12}(C)\ w_{12}(C)\ u_2(C)] = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 8 & 3 \\ 1 & -1 & 0 & -1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

with

$$R^{-1} = \begin{pmatrix} 3 & 1 & -1 & -2 \\ -20 & -7 & 11 & 16 \\ -9 & -3 & 5 & 7 \\ 23 & 8 & -13 & -18 \end{pmatrix}.$$

The Jordan canonical form of C is as follows:

$$J(C) = R^{-1}CR = \left(\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ \hline 0 & 0 & 0 & 4 \end{array}\right).$$

Case 3. The rank of the matrix $\widehat{D} = D - 4I_4 = \begin{pmatrix} -11 & -4 & 6 & 9 \\ -11 & -4 & 6 & 9 \\ -11 & -4 & 6 & 9 \\ -11 & -4 & 6 & 9 \end{pmatrix}$ is one, so there are three blocks in the Jordan normal form. The three linearly independent eigenvectors of D are as follows:

$$u_1(D) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; u_2(D) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \text{ and } u_3(D) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -2 \end{pmatrix}$$

The generalized eigenvector $v_1(D) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is a solution of the system $\widehat{D}v = u_1(D)$.

Note that the eigenvectors $\begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 3 \\ -2 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 4 \\ 0 \\ 3 \end{pmatrix}$ are linearly independent but do not produce any generalized eigenvector since all the rows of $\widehat{D}$ are identical but components of these three vectors are not all equal. The matrix

$$S = [u_1(D) \; v_1(D) \; u_2(D) \; u_3(D)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & -1 & 0 & -2 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -11 & -4 & 6 & 9 \\ -4 & -1 & 2 & 3 \\ 6 & 2 & -3 & -5 \end{pmatrix} \qquad \text{with}$$

will produce the Jordan normal form as follows:

$$J(D) = S^{-1}DS = \left[\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right].$$

Remark 1. The following vectors

$$u_1'(D) = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}, \; u_2(D) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \text{ and } u_3(D) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -2 \end{pmatrix}$$

are also linearly independent eigenvectors of the matrix D; but none of them could produce a generalized eigenvector v which is needed to obtain $J(D)$. Since $\widehat{D}^2$ is the zero matrix, it follows that any vector is a generalized eigenvector of D; for example the first three columns of the invertible matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix}$$

are the eigenvectors of D, and the forth column is a generalized eigenvector in the generalized sense. Unfortunately

$$T^{-1}DT = \begin{pmatrix} 4 & 0 & 0 & 9 \\ 0 & 4 & 0 & -9 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

is not the Jordan canonical form of D. In this case, we need to go back to our eigenvectors and choose the ones that will produce a Jordan canonical form.

Example. Consider the matrix

$$M = \begin{pmatrix} 7 & 4 & 3 & -7 \\ 3 & 5 & 2 & -5 \\ 1 & 0 & 2 & 0 \\ 5 & 4 & 3 & -5 \end{pmatrix}$$

with the characteristic polynomial

$$K_M(\lambda) = \lambda^4 - 9\lambda^3 + 29\lambda^2 - 39\lambda + 18.$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2,$, and $\lambda_3 = \lambda_4 = 3$. with the corresponding eigenvectors:

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad u_3 = u_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

The equation $(M - 3\,I_4)v = u_3$ produces the generalized eigenvector $v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$. Thus we have

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{with} \quad P^{-1} = \begin{pmatrix} -1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 2 & 1 & -3 \end{pmatrix}.$$

Hence the Jordan canonical form $J(M)$ of M is obtained as follows:

$$\begin{aligned} P^{-1}\,M\,P &= \begin{pmatrix} -1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 7 & 4 & 3 & -7 \\ 3 & 5 & 2 & -5 \\ 1 & 0 & 2 & 0 \\ 5 & 4 & 3 & -5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} = J(M). \end{aligned}$$

♣ **System of Linear Differential Equations**. Consider the system

$$X'(t) = A\,X(t) \quad \text{with} \quad P^{-1}AP = J(A).$$

Let $X(t) = P\,Y(t)$, then $X'(t) = P\,Y(t)$, $P^{-1}X(t) = Y(t)$, and $P^{-1}X'(t) = Y('t)$. We have:

$$Y'(t) = P^{-1}X'(t) = P^{-1}A\,X'(t) = P^{-1}A\,P\,Y(t) = J(A)\,Y(t).$$

Thus by solving $Y'(t) = J(A)\,Y(t)$, we may obtain the solution of the original system as follows:

$$X'(t) = PY'(t) = P\,[J(A)Y(t)] = P\left[P^{-1}A\,P\,Y(t)\right] = \left[P\,P^{-1}\right]A\,P\,Y(t) = A\,[P\,Y(t)] = A\,X(t)\,.$$