Laplace Transforms

A real function $F(t)$ is said to be of *Class A*, if $F(t)$ is *Sectionally Continuous* for $t > 0$, and

$$\lim_{t\to\infty} e^{-t^r} F(t) = 0 \quad (r > 1).$$

If $F(t)$ is a function of *Class A*, then for some s, we define the *Laplace Transform* of $F(t)$ denoted by $\mathcal{L}_s\{F(t)\}$ as follows:

$$\mathcal{L}_s\{F(t)\} = \int_0^\infty e^{-st} F(t) dt.$$

The Laplace Transform is a *one to one* function, therefore its *inverse function* is also a function and is called the *Inverse Transform* denoted by $\mathcal{L}_t^{-1}\{f(s)\}$; so

$$F(t) = \mathcal{L}_t^{-1}\left\{\mathcal{L}_s\{F(t)\}\right\}.$$

(1)	$s > a$	$\mathcal{L}_s\{F(t)e^{at}\} = L_{s-a}\{F(t)\}$
(2)	$n \in I\!N$	$\mathcal{L}_s\{t^n\} = \frac{n!}{s^{n+1}}$
(3)	$n = 2p + 1$	$\mathcal{L}_s\{t^{\frac{n}{2}}\} = \frac{1}{s^{p+1}}\left(\frac{n}{2}\frac{n-2}{2}\dots\frac{3}{2}\frac{1}{2}\sqrt{\frac{\pi}{s}}\ \right)$

By differentiating $F(t)$, N times and then by taking the Laplace Transform, we have:

$$\boxed{(4) \qquad \mathcal{L}_s\{F^{(N)}(t)\} = s^N \mathcal{L}_s\{F(t)\} - s^{N-1}F(0) - s^{N-2}F'(0)\dots F^{N-1)}(0)}$$

By differentiating N times the Laplace Transform of $F(t)$ with respect to the parameter s, we obtain

$$\boxed{(5) \qquad \frac{d^N}{ds^N}\left(\mathcal{L}_s\{F(t)\}\right) = \mathcal{L}_s\{(-t)^N F(t)\}}$$

An immediate consequence of the above formula is:

$$\boxed{(6) \qquad \mathcal{L}_s\{t^N F(t)\} = (-1)^N \frac{d^N}{ds^N}\left(\mathcal{L}_s\{F(t)\}\right)}$$

By integrating the Laplace Transform of $F(t)$, we obtain:

$$\boxed{(7) \qquad \mathcal{L}_s\left\{\frac{F(t)}{t}\right\} = \int_s^\infty L_r\{F(t)\}dr}$$

A function $F(t)$ is called *Periodic of Period* ω , if for all t in the domain of $F(t)$,

$$F(t+\omega) = F(t).$$

$$\boxed{(8) \qquad F(t+\omega) = F(t) \qquad \mathcal{L}_s\{F(t)\} = \frac{1}{1-e^{-s\omega}}\int_0^\omega e^{-su}F(u)du}$$

♣ **Gamma Function**. For $x > 0$, we define the Gamma function as follows:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt = \mathcal{L}_s\{t^{x-1}\}.$$

$$\boxed{\begin{array}{lll} (9) & s > 0 & \Gamma(x+1) = x\Gamma(x) = s^{x+1}\mathcal{L}_s\{t^x\} \\ (10) & n \in \mathbb{N} & \Gamma(n+1) = n\Gamma(n) = n! \\ (11) & & \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{array}}$$

♣ **A Step Function**. We define the step function $\alpha(t)$ as follows:

$$\alpha(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{otherwise.} \end{cases}$$

Thus

$$\alpha(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{otherwise;} \end{cases} \qquad \alpha(t-a)F(t) = \begin{cases} 0, & \text{if } t < a \\ \text{F(t)}, & \text{otherwise;} \end{cases}$$

and

$$[\alpha(t-a) - \alpha(t-a)]\,F(t) = \begin{cases} \text{F(t)}, & \text{if } a < t < b \\ 0, & \text{otherwise.} \end{cases}$$

By using the α function, we obtain:

$$\boxed{(12) \quad \mathcal{L}_s\{\alpha(t-a)F(t-a)\} = e^{-as}\mathcal{L}_s\{F(t)}$$

♣ **The Convolution Theorem**. If $f(s) = \mathcal{L}_s\{F(t)\}$ and $g(s) = \mathcal{L}_s\{G(t)\}$, then

$$\boxed{(13)\quad \mathcal{L}_t^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = \int_0^t F(t-u)G(u)du}$$

Since $\mathcal{L}_s\{\alpha(t)\} = \dfrac{1}{s}$, it follows that

$$\boxed{(14)\quad \mathcal{L}_t^{-1}\left\{\tfrac{f(s)}{s}\right\} = \int_0^t F(u)du}$$

♣ **Trigonometric and Hyperbolic Functions**. Finally we give some usefull formulas for the inverse transform.

$$\boxed{\begin{array}{ll}
\mathcal{L}_t^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt) & \mathcal{L}_t^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt) \\
\mathcal{L}_t^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh(kt) & \mathcal{L}_t^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt) \\
\mathcal{L}_t^{-1}\left\{\arctan(\frac{k}{s})\right\} = \sin(kt)/t & \mathcal{L}_t^{-1}\left\{\ln\left(\frac{s+k}{s-k}\right)\right\} = 2\sinh(kt)/t
\end{array}}$$

Laplace Transform Solution of Linear Differential Equations with Constant Coefficients

We now consider how the Laplace Transform may be applied to solve the initial-value problem consisting of the nth-order linear differential equation with constant coefficients.

Consider the initial-value problem:

$$\begin{cases} a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = b \\ y(0) = c_0, y'(0) = c_1, \ldots, y^{(n-1)}(0) = c_{n-1}. \end{cases}$$

By taking first, the Laplace Transform of both sides of the equation and then the Inverse Transform of the resulting equation, we may find a solution to the problem. We illustrate our method with the following example. We shall denote $\mathcal{L}_s\{x\}$ by $X(s)$ and $\mathcal{L}_s\{y\}$ by $Y(s)$.

Solve subject to:	$y'' - 6y' + 9y = t^2e^3t$ $y(0) = 2, \quad y'(0) = 6.$
Step 1.	$\mathcal{L}_s\{y''\} - 6\mathcal{L}_s\{y'\} + 9\mathcal{L}_s\{y\} = \mathcal{L}_s\{t^2e^3t\}$
Step 2.	$\left[s^2Y(s) - 2s - 6\right] - 6\left[sY(s) - 2\right] + 9Y(s) = \frac{2}{(s-3)^3}$
Step 3.	$\left[s^2 - 6s + 9\right]Y(s) = 2(s-3) + \frac{2}{(s-3)^3}$
Step 4.	$Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5}$
Solution:	$y(t) = L_t^{-1}\{Y(s)\} = 2e^3t + \frac{1}{12}t^4e^3t$

Finally, we find the solution of a system of linear differential equations with initial conditions.

Solve	$\begin{cases} x' - x - y = 2e^t, \\ y' + 2x' - y = e^t, \end{cases}$
subject to:	$x(0) = 0, \quad y(0) = 0.$
Step 1.	$\begin{cases} \mathcal{L}_s\{x'\} - \mathcal{L}_s\{x\} - \mathcal{L}_s\{y\} = \mathcal{L}_s\{2e^t\} \\ \mathcal{L}_s\{y'\} + 2\mathcal{L}_s\{x'\} - \mathcal{L}_s\{y\} = \mathcal{L}_s\{e^t\} \end{cases}$
Step 2.	$\begin{cases} (s-1)X(s) - Y(s) = \dfrac{2}{s-1} \\ 2sX(s) + (s-1)Y(s) = \dfrac{1}{s-1} \end{cases}$
Step 3.	$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ 2s & s-1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{s-1} \\ \frac{1}{s-1} \end{bmatrix}$
Step 4.	$\begin{cases} X(s) = \dfrac{2s-1}{(s-1)(s^2+1)} \\ Y(s) = -\dfrac{3s+1}{(s-1)(s^2+1)} \end{cases}$
Solutions:	$\begin{cases} x(t) = \mathcal{L}_t^{-1}\{X(s)\} = \dfrac{1}{2}\left[e^t - \cos t + 3\sin t\right] \\ y(t) = \mathcal{L}_t^{-1}\{Y(s)\} = -\left[2e^t - 2\cos t + \sin t\right] \end{cases}$