**Direct Factorization of Matrices**

We discuss the factoring of a square matrix $A$ in terms of a lower-triangular matrix $L$ and an upper-triangular matrix $U$. It is known that this factorization exists whenever the linear system $Ax = b$ can be solved uniquely by Gaussian elimination without row interchanges. The system $LUx = Ax = b$ can then be transformed into the system $Ux = L^{-1}b$ and, since $U$ is upper triangular, backward substitution can be applied.

♣ **Doolittle's Decomposition**. This method allows us to factor a square matrix $A$ into $LU$, where $L$ is a lower-triangular matrix with ones on the main diagonal and $U$ an upper-triangular matrix with nonzero diagonal entries.

**ALGORITHM.**

Input a square matrix A.
$n = size(A);\ m = n(1);$ ( Find the size of A.)
$U = zeros(m);$ ( Define $U$ as an $m \times m$ zero matrix.)
$L = eye(m);$ ( Define $L$ as an $m \times m$ identity matrix.)
$U(1, 1 : m) = A(1, 1 : m),\ L(2 : m, 1) = (1/U(1, 1)) * A(2 : m, 1)$
$for\ k = 2 : m$
$\quad for\ j = k : m,\ U(k, j) = A(k, j) - L(k, 1 : k-1) * U(1 : k-1, j);\ end;\ U$
$\quad for\ i = k+1 : m,\ L(i, k) = (A(i, k) - L(i, 1 : k) * U(1 : k, k))/U(k, k);\ end;\ L$
$end$

**Example.** Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 5 & 6 & 8 & 8 \\ 1 & 3 & 6 & 5 \end{pmatrix} = \begin{bmatrix} A(1,1:4) \\ A(2,1:4) \\ A(3,1:4) \\ A(4,1:4) \end{bmatrix} = [A(i,j)].$

First, we initialize $U$ and $L$.

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first row of $U$ is : $U(1, 1 : 4) = A(1, 1 : 4)$.
The first column of $L$ is : $L(2 : 4, 1) = (1/U(1, 1)) * A(2 : 4, 1)$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The second row of $U$ is : $for\ j = 2 : 4,\ U(2, j) = A(2, j) - L(2, 1 : 1) * U(1 : 1, j);\ end.$

The second column of $L$ is: $for\ i = 3:4,\ L(i,2) = (A(i,2) - L(i,1:2) * U(1:2,2))/U(2,2);\ end.$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

The third row of $U$ is : $for\ j = 3:4,\ U(3,j) = A(3,j) - L(3,1:2) * U(1:2,j);\ end.$

The third column of $L$ is: $for\ i = 3:4,\ L(i,3) = (A(i,3) - L(i,1:3) * U(1:3,3))/U(3,3);\ end.$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

The last row of $U$ is : $for\ j = 4:4,\ U(4,j) = A(4,j) - L(4,1:3) * U(1:3,j);\ end.$

The last column of $L$ is: $for\ i = 4:4,\ L(i,4) = (A(i,4) - L(i,1:4) * U(1:4,4))/U(4,4);\ end.$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

**Remark:** Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ and the linear system $Ax = b$. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & | & b_1 \\ 1 & 1 & 2 & | & b_2 \\ 1 & 2 & 1 & | & b_3 \end{bmatrix}.$$

By using the Gaussian elimination, we obtain:

$$\begin{bmatrix} 1 & 1 & 1 & | & b_1 \\ 0 & 0 & 1 & | & b_2 - b_1 \\ 0 & 1 & 0 & | & b_3 - b_1 \end{bmatrix}.$$

At this point we must interchange the second and third row. Now suppose $A = L * U$, then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

We have $$u_{11} = u_{12} = u_{13} = 1, \quad l_{21} = l_{31} = 1;$$

then $u_{22} = 0$, which makes

$$2 = a_{32} = l_{31}u_{12} + l_{32}u_{22} = 1 \times 1 + l_{32} \times 0 = 1$$

a contradiction. Thus $A$ may not be expressed as a product of a lower triangular matrix $L$ and an upper triangular matrix $U$.

♣ **Choleski's Decomposition**. If $A$ is a positive definite symmetric matrix , then $A$ has a factorization of the form $LL^t$, where $L$ is a lower-triangular matrix. This factorization is known as the Choleski's decomposition method.

Consider the positive definite symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots\cdots & & & a_{1n} \\ a_{21} & a_{22} \cdots\cdots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} \cdots\cdots & & & a_{nn} \end{pmatrix}.$$

Then let

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & \cdots\cdots & & 0 \\ a_{21} & 0 & \ddots & \cdots & \vdots \\ a_{31} & a_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{A}^1 \\ \widetilde{A}^2 \\ \vdots \\ \widetilde{A}^n \end{pmatrix} = [\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_n].$$

In order to express $A = LL^t$, we define the diagonal matrix

$$D = \begin{pmatrix} d_{11} & 0 & \cdots\cdots & & 0 \\ 0 & d_{22} & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots\cdots & & & d_{nn} \end{pmatrix}$$

and the lower triangular matrix

$$\widetilde{L} = \begin{pmatrix} 0 & 0 & \cdots\cdots & & 0 \\ \widetilde{l}_{21} & 0 & \ddots & \cdots & \vdots \\ \widetilde{l}_{31} & \widetilde{l}_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \widetilde{l}_{n1} & \widetilde{l}_{n2} & \cdots & \widetilde{l}_{nn-1} & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{L^1} \\ \widetilde{L^2} \\ \vdots \\ \widetilde{L^n} \end{pmatrix} = [\widetilde{L_1}, \widetilde{L_2}, \cdots, \widetilde{L_n}].$$

Then we use the following procedure:

Step 1. $d_{11} = \sqrt{a_{11}}$ and $\widetilde{L_1} = \dfrac{1}{d_{11}}\widetilde{A}_1$ .

Step 2. $d_{22} = \sqrt{a_{22} - \widetilde{L}^2\widetilde{L}^2}$ and $\widetilde{l_{i2}} = \dfrac{a_{i2} - \widetilde{L}^i\widetilde{L^2}}{d_{22}}$, where $i = 3, 4, \cdots, n$.

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Step j. $d_{jj} = \sqrt{a_{jj} - \widetilde{L}^j\widetilde{L}^j}$ and $\widetilde{l}_{ij} = \dfrac{a_{ij} - \widetilde{L}^i\widetilde{L}^j}{d_{jj}}$, where $i = j+1, j+2, \cdots, n$.

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Finally, $d_{nn} = \sqrt{a_{nn} - \widetilde{L}^n\widetilde{L}^n}$ .

The lower triangular matrix $L = \widetilde{L} + D$ is the desired matrix.

**ALGORITHM.**

Input a positive definite symmetric matrix A.

$n = size(A);\ m = n(1);$ % Find the size of A.

$D = zeros(m);$ % Define the diagonal matrix $D$ as an $m \times m$ zero matrix.

$L = zeros(m);$ % Define $L$ as an $m \times m$ zero matrix.

$D(1,1) = sqrt(A(1,1)),\ L(2:m,1) = (1/D(1,1)) * A(2:m,1)$

$fork = 2 : m$

$\quad for\ j = k : m,\ D(k,k) = sqrt(A(k,k) - L(k,1:k) * L(k,1:k)^t);\ end,\ D$

$\quad for\ i = k+1 : m,\ L(i,k) = (A(i,k) - L(i,1:k) * L(k,1:k)^t)/D(k,k);\ end;\ L$

$end$

$D(m,m) = sqrt(A(m,m) - L(m,1:m) * L(m,1:m)^t)$

$L = L + D$

**EXAMPLE 1.** Let $A = \begin{pmatrix} 9 & 3 & 3 \\ 3 & 5 & -1 \\ 3 & -1 & 18 \end{pmatrix}$. Then $\widetilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & -1 & 0 \end{pmatrix}$.

1 ) $d_{11} = \sqrt{9} = 3$ *and* $\widetilde{L}_1 = 1/3 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. So,

$$\widetilde{L}^2 = (1,0,0) \quad and \ \widetilde{L}^3 = (1, \tilde{l}_{32}, 0) \ .$$

2 ) $d_{22} = \sqrt{5 - (1,0,0).(1,0,0)} = 2,\ \tilde{l}_{32} = \frac{1}{2}[-1 - (1,0,0).(1,\tilde{l}_{32},0)] = -1.$ So,

$$\widetilde{L}^3 = (1,-1,0) \quad and \ \ \widetilde{L}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} .$$

3 ) Finally, $d_{33} = \sqrt{18 - (1,-1,0).(1,-1,0)} = \sqrt{16} = 4$ . Thus

$$L = \widetilde{L} + D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 4 \end{pmatrix} .$$

**Example 2.** Consider the positive definite symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 13 & 18 & 23 \\ 3 & 18 & 50 & 62 \\ 4 & 23 & 62 & 126 \end{pmatrix} = \begin{bmatrix} A(1,1:4) \\ A(2,1:4) \\ A(3,1:4) \\ A(4,1:4) \end{bmatrix} = [A(i,j)].$$

We initialize $D$ and $L$.

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \widetilde{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first diagonal entry of $D$ is : $D(1,1) = \sqrt{A(1,1)}$.

The first column of $\widetilde{L}$ is : $\widetilde{L}(2:4,1) = (1/D(1,1)) * A(2:4,1)$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \widetilde{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

The second diagonal entry of $D$ is : $D(2,2) = \sqrt{A(2,2) - \widetilde{L}(2,1:1) * \widetilde{L}(2,1:1)^t}$.

The second column of $\widetilde{L}$ is: $for\ i = 3:4,\ \widetilde{L}(i,2) = (A(i,2) - \widetilde{L}(i,1:2) * \widetilde{L}(2,1:2))/D(2,2);\ end.$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \widetilde{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 5 & 0 & 0 \end{bmatrix}$$

The third diagonal entry of $D$ is : $D(3,3) = \sqrt{A(3,3) - \widetilde{L}(3,1:2) * \widetilde{L}(3,1:2)^t)}$. The third column of $\widetilde{L}$ is: $for\ i = 3:4,\ \widetilde{L}(i,3) = (A(i,3) - \widetilde{L}(i,1:3) * \widetilde{L}(3,1:3)^t)/D(3,3);\ end.$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \widetilde{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 5 & 6 & 0 \end{bmatrix}$$

The last diagonal entry of $D$ is : $D(4,4) = \sqrt{A(4,4) - \widetilde{L}(4,1:3) * \widetilde{L}(4,1:3)^t}$.

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

The matrix $L = D + \widetilde{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 5 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 4 & 5 & 0 \\ 4 & 5 & 6 & 7 \end{bmatrix}.$

**Remark:** All the principal sub-matrices of a positive definite matrix must be invertible. Consider the symmetric matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ with $\det(A) = -1$. $A$ has at least one non-invertible principal sub-matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ which makes $A$ non-positive definite. Now suppose $A = L * L^t$, then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}.$$

We have $\qquad l_{11} = l_{21} = l_{31} = 1, \quad l_{21} = l_{31} = 1,$ and $l_{22} = 0$

which makes $\det(A) = 0$, a contradiction.

♣ **Inverse of Bidiagonal Matrices**. Consider a $n \times n$ non singular lower-triangular matrix $A$ where all the diagonals are zero, except the main diagonal $D$, and the $L$ diagonal below that. The following algorithm explains how the inverse of $A$ is obtained.

We explain the method using an example:

**Example.** Consider the matrix

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & -5 & 4 \end{pmatrix} = (a_{ij}).$$

If $B = (b_{ij}) = A^{-1}$, then we have:

$$b_{1,1} = (-1)^0 \frac{1}{a_{1,1}} = \frac{1}{4}, \quad b_{2,2} = (-1)^0 \frac{1}{a_{2,2}} = \frac{1}{3}, \quad b_{3,3} = (-1)^0 \frac{1}{a_{3,3}} = -1, \quad b_{4,4} = (-1)^0 \frac{1}{a_{4,4}} = \frac{1}{4}$$

$$b_{2,1} = (-1)^1 \frac{a_{2,1}}{a_{1,1} * a_{2,2}} = \frac{1}{6}, \quad b_{3,2} = (-1)^1 \frac{a_{3,2}}{a_{2,2} * a_{3,3}} = -1, \quad b_{4,3} = (-1)^1 \frac{a_{4,3}}{a_{3,3} * a_{4,4}} = -\frac{5}{4}$$

$$b_{3,1} = (-1)^2 \frac{a_{2,1} * a_{3,2}}{a_{1,1} * a_{2,2} * a_{3,3}} = -\frac{1}{2}, \quad b_{4,2} = (-1)^2 \frac{a_{3,2} * a_{4,3}}{a_{2,2} * a_{3,3} * a_{4,4}} = -\frac{5}{4}$$

$$b_{4,1} = (-1)^3 \frac{a_{2,1} * a_{3,2} * a_{4,3}}{a_{1,1} * a_{2,2} * a_{3,3} * a_{4,4}} = -\frac{5}{8}$$

Hence the inverse of $A$ is

$$A^{-1} = B = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{2} & -1 & -1 & 0 \\ -\frac{5}{8} & -\frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{pmatrix}$$

♣ **Crout's Reduction For Tridiagonal Linear Systems**. This is an algorithm for solving an $n \times n$ system of linear equations whose coefficient matrix is tridiagonal.

Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & \vdots \\ \vdots & a_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

is a non singular tridiagonal matrix.

First we obtain $A = LU$, where

$$L = \begin{pmatrix} l_{11} & 0 & \cdots\cdots & & 0 \\ l_{21} & l_{22} & 0 & \cdots & \vdots \\ \vdots & l_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & & l_{n,n-1} & l_{nn} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & u_{12} & \cdots\cdots & & 0 \\ 0 & 1 & u_{23} & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & & 0 & 1 \end{pmatrix},$$

then we solve the system $Lz = b$, and finally we obtain $x$ by solving $Ux = z$.

**ALGORITHM.**

Input the tridiagonal matrix $A$ and the vector $b$.

$n = size(A); m = n(1);$ % Find the size of $A$.

$L = zeros(m);$ % Define $L$ as an $m \times m$ zero matrix.

$U = eye(m);$ % Define $U$ as an $m \times m$ identity matrix.

$z = zeros(m, 1);\ \ x = zeros(m, 1)$ % $z$ and $x$ are zero vectors.

$for\ \ i = 2 : m,\ \ L(i, i-1) = A(i, i-1);\ \ end;$

$L(1, 1) = A(1, 1),\ \ U(1, 2) = (1/L(1, 1)) * A(1, 2)$

$for\ \ k = 2 : m,$

  $for\ \ j = k : m,$

    $L(k, k) = A(k, k) - L(k, k-1) * U(k-1, k);$

    $U(k, k+1) = A(k, k+1)/L(k, k);$

  $end;\ \ L,\ \ U$

$end$

$L(m, m) = A(m, m) - L(m, m-1) * U(m-1, m);\ \ L$

$z(1) = b(1)/L(1, 1);$

$for\ \ k = 2 : m,\ \ z(k) = (1/L(k, k)) * (b(k) - L(k, k-1) * z(k-1));\ \ end;\ \ z$

$x(m) = z(m);$

$for\ \ k = m-1 : -1 : 1,\ \ x(k) = z(k) - U(k, k+1) * x(k+1);\ \ end;\ \ x$

$end$

**Example.** Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & 2 & -6 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 3 & -4 \end{pmatrix} = \begin{bmatrix} A(1, 1:4) \\ A(2, 1:4) \\ A(3, 1:4) \\ A(4, 1:4) \end{bmatrix} = [A(i, j)] \qquad b = \begin{pmatrix} 2 \\ 4 \\ -10 \\ -4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

We initialize $L$, $U$, $x$ and $z$.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We have :

$for\ i = 2:m,\ L(i,i-1) = A(i,i-1);\ end;$

$L(1,1) = A(1,1),\ U(1,2) = (1/L(1,1)) * A(1,2)$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$L(2,2) = A(2,2) - L(2,1) * U(1,2);$

$U(2,3) = A(2,3)/L(2,2);$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$L(3,3) = A(3,3) - L(3,2) * U(2,3);$

$U(3,4) = A(3,4)/L(3,3);$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$L(4,4) = A(4,4) - L(4,3) * U(3,4);$ thus:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$Ax = b \quad \Rightarrow \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -10 \\ -4 \end{bmatrix}.$$

Let

$$z = L^{-1}b = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ \frac{1}{12} & -\frac{1}{6} & \frac{1}{4} & 0 \\ -\frac{1}{20} & \frac{1}{10} & -\frac{3}{20} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}.$$

Then

$$x = U^{-1}z = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$