

First Order Differential Equations

Definitions and Notations. The k -th order derivative of the function $y(x)$ is denoted by $D_x^k y$ or simply $D^k y$. Thus

$$D^k y = D_x^k y = \frac{d^k y}{dx^k} \quad \text{and} \quad D_x^0 y = Iy = y.$$

A general n -th order, ordinary differential equation is represented by

$$F(x, y, Dy, \dots, D^n y) = 0;$$

so an ordinary differential equation is an equation (E) which contains terms such as $D^k y$. The highest power of D in (E) is called the order of the equation.

The equation $F(x, D)y = R(x)$, where

$$F(x, D) = a_0(x)D^n + a_1(x)D^{n-1} + \dots + a_n(x)I,$$

is said to be *linear of order n* . When $R(x) = 0$, then the linear differential equation is called *homogeneous*.

If a solution of $F(x, y, D) = 0$ can be expressed as $y = f(x)$ (i.e., y is a function of x), then this solution is called an *explicit solution*. If we obtain $f(x, y) = 0$ as a solution of our differential equation, then we say that only an *implicit solution* has been found.

A first order differential equation may be expressed as follows:

$$\frac{dy}{dx} = f(x, y).$$

The problem

$$\begin{cases} \text{Solve:} & \frac{dy}{dx} = f(x, y) \\ \text{Subject to:} & y(x_0) = y_0 \end{cases} \quad (1)$$

is called an *initial-value problem*. The first equation gives the slope of the curve y at any point x , and the second equation specifies one particular value of the function $y(x)$.

Existence. Will every initial-value problem have a solution? No, some assumptions must be made about $f(x, y)$, and even then we can only expect the solution to exist in a neighborhood of $x = x_0$. As an example of what could happen, consider

$$\begin{cases} \frac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$$

The solution curve starts at $x = 0$ with slope one; that is, $y'(0) = 1$. Since the slope is positive, $y(x)$ is *increasing* near $x = 0$. Therefore, the expression $1 + y^2$ is also increasing. Hence, y' is increasing. Since y and y' are both increasing and are related by the equation $y' = 1 + x^2$, we can expect that at some finite value of x there will be no solution; that is, $y(x) = +\infty$. As a matter of fact, this occurs at $x = \pi/2$ because the *analytic solution* of the initial-value problem is $y(x) = \tan x$.

Theorem 1. If $f(x, y)$ is continuous in a rectangle R centered at (x_0, y_0) , say

$$R = \{(x, y) : |x - x_0| \leq \alpha, \quad |y - y_0| \leq \beta\} \quad (2)$$

then the initial-value problem (1) has a solution $y(x)$ for $|x - x_0| \leq \min(\alpha, \beta/M)$, where M is the maximum of $|f(x, y)|$ in the rectangle R .

Uniqueness. It can happen, even if $F(x, y)$ is continuous, that the initial-value problem does not have a unique solution. A simple example of this phenomenon is given by the problem

$$\begin{cases} \frac{dy}{dx} = y^{2/3} \\ y(0) = 0 \end{cases}$$

It is obvious that the zero function, $y(x) \equiv 0$, is a solution of this problem. Another solution is the function

$$y(x) = \frac{1}{27}x^3$$

To prove that the initial-value problem (1) has a *unique solution* in a neighborhood of $x = x_0$, it is necessary to make some assumptions about $f(x, y)$. Here are the usual theorems used for the uniqueness of the solution of an initial-value problem..

Theorem 2. If $f(x, y)$ and $\frac{\partial f(x, y)}{\partial y}$ are continuous in a rectangle R defined by (2), then the initial-value problem (1) has a unique solution in the interval $|x - x_0| \leq \min(\alpha, \beta/M)$.

Theorem 3. If $f(x, y)$ is continuous in the strip

$$a \leq x \leq b, \quad -\infty < y < \infty$$

and satisfies the inequality:

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|, \quad (3)$$

then the initial-value problem (1) has a unique solution in the interval $[a, b]$.

The Inequality (3) is called a *Lipschitz condition* for the second variable y . We see immediately that this condition is *stronger* than the continuity.

Definition. Let $f(u)$ be a function where $u \in \mathbb{R}^n$ such that $f(\lambda u) = \lambda^k f(u)$, for a suitable real λ . Then $f(u)$ is called a *homogeneous function of degree k* .

♠ **Note.** Do not confound homogeneous equations with homogeneous functions.

First Order Differential Equations. There are several classes of differential equations of order one. We shall explain how to classify and solve some of these classes.

♣ **Separation of Variables.** If the equation $M(x, y)dx + N(x, y)dy = 0$ can be changed into $M(x)dx + N(y)dy = 0$, then we may change it into $M(x)dx = -N(y)dy$. The equation will be solved by integrating both sides of the equality.

Equation:	$2xydx - (x^2 + 1)dy = 0$
Step 1.	$[2x/(x^2 + 1)]dx = dy/y$
Step 2.	$\int [2x/(x^2 + 1)]dx = \int dy/y$
Implicit Solution:	$\ln(x^2 + 1) = \ln y/C $
Explicit Solution:	$y = C(x^2 + 1)$

♣ **Exact Equation.** The equation

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be exact, if $M_y(x, y) = N_x(x, y)$. Since $F_{xy}(x, y) = F_{yx}(x, y)$ for any smooth function $F(x, y)$, we conclude that there exists a constant function $C = F(x, y)$ such that

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y(x, y) = N(x, y)$$

with

$$0 = dC = dF(x, y) = M(x, y)dx + N(x, y)dy.$$

Algorithm. If an equation is exact, then let

$$C = F(x, y) = \int M(x, y)dx + \phi(y) \tag{4}$$

$$C = F(x, y) = \int N(x, y)dy + \psi(x). \tag{5}$$

We select (4) or (5), whichever is easier to integrate.

By setting $\partial F(x, y)/\partial y = N(x, y)$ (or $\partial F(x, y)/\partial x = M(x, y)$) we obtain $\phi'(y)$ or $\psi'(x)$. To find $\phi(y)$ or $\psi(x)$, we just integrate $\phi'(y)$ or $\psi'(x)$. The solution of the equation is then obtained from (4) or (5).

Equation:	$(x + 2y)dx + (2x + y)dy = 0$
Step 1.	$M_y(x, y) = N_x(x, y) = 2$
Step 2.	$C = F = \int M(x, y) dx = \int (x + 2y)dx + \phi(y) \Rightarrow F = x^2/2 + 2xy + \phi(y)$
Step 3.	$0 + 2x + \phi'(y) = N(x, y) = 2x + y \Rightarrow \phi(y) = \int \phi'(y) dy = \int y dy = y^2/2$
Implicit Solution:	$C = x^2/2 + 2xy + y^2/2 \quad \text{or} \quad x^2 + 4xy + y^2 = C$

◇ **Integrating Factor.** If by multiplying the equation

$$M(x, y)dx + N(x, y)dy = 0 \quad (E)$$

by a function $u(x)$, $u(y)$, or $u(x, y)$; we may be able to solve the new equation. Then $u(x)$, $u(y)$, or $u(x, y)$ is called an integrating factor.

The following functions make the equation (E) *Exact*:

1. If

$$\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} \quad \text{or} \quad -\frac{[M_y(x, y) - N_x(x, y)]}{M(x, y)}$$

is a function of x (or y). Then the integrating factor is:

$$u(x) = \exp\left(\int \frac{M_y(x, y) - N_x(x, y)}{N} dx\right) \quad \text{or} \quad v(y) = \exp\left(-\int \frac{M_y(x, y) - N_x(x, y)}{N} dy\right).$$

2. If the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions of the same degree and $xM(x, y) + yN(x, y) \neq 0$, then

$$u(x, y) = \frac{1}{xM(x, y) + yN(x, y)}$$

is an integrating factor which makes the equation Exact.

♣ **Homogeneous Functions.** If in the equation

$$M(x, y)dx + N(x, y)dy = 0, \quad (E)$$

where $M(x, y)$ and $N(x, y)$ are both homogeneous functions of the same degree. Then by using a substitution we may solve the equation by the method of *separation of variables*.

Algorithm. Set $y = ux$ (or $x = vy$) in (E), then $dy = xdu + udx$ (or $dx = ydv + vdy$). We then obtain

$$\widehat{M}(x, u)dx + \widehat{N}(x, u)du = 0 \quad \text{or} \quad \widehat{M}(v, y)dv + \widehat{N}(v, y)dy = 0$$

which can be solved by using separation of variables.

Equation:	$(x^2 + 2y^2)dx - xydy = 0$
Step 1.	$M(\lambda x, \lambda y) = \lambda^2 M(x, y)$ and $N(\lambda x, \lambda y) = \lambda^2 N(x, y)$ So $M(x, y)$ and $N(x, y)$ are both homogeneous of the same degree.
Step 2.	Let $y = xu$, then $dy = xdu + udx$
Step 3.	$(x^2 + 2x^2u^2)dx - x^2u(xdu + udx) = 0$
Step 4.	$(x^2 + x^2u^2)dx = x^3u du \Rightarrow (1 + u^2)dx = xu du$
Step 5.	$\int [1/x] dx = \int [u/(1 + u^2)] du \Rightarrow \ln(Cx) = \frac{1}{2} \ln(1 + u^2)$
Implicit Solution:	$Cx^2 = 1 + [x/y]^2 \Rightarrow Cx^4 - x^2 = y^2$

Consider the equation:

$$(a_1x + b_1y + c_1) dx + (a_2x + b_2y + c_2) dy = 0. \quad (E)$$

If the point (h, k) is a solution to the linear system

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0. \end{cases}$$

Then by setting $x = u + h$ and $y = v + k$ in (E) , we obtain the equation

$$(a_1u + b_1v)du + (a_2u + b_2v)dv = 0$$

which has homogeneous coefficients of degree one.

If $(a_1x + b_1y + c_1) = s(a_2x + b_2y + c_2) + r$, then by substituting u for $(a_1x + b_1y + c_1)$ and eliminating x or y , we may solve the equation by separation of variables.

♣ **Linear Equation.** Consider the first order linear differential equation:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

By multiplying the equation by the integrating factor

$$u(x) = \exp\left(\int P(x)dx\right),$$

the equation changes into

$$u(x) dy + y[P(x)u(x) dx] = u(x)Q(x) dx.$$

Notice that

$$P(x)u(x) dx = P(x) \exp\left(\int P(x)dx\right) = d[u(x)].$$

Thus the new equation is:

$$d[yu(x)] = u(x)Q(x) dx.$$

By integrating both sides of the equation, we obtain

$$yu(x) = \int u(x)Q(x) dx.$$

♡ **Bernoulli's Equation.** The following equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (n \neq 1)$$

is called a Bernoulli equation. For $n = 1$, the equation may be solved using the separation of variables.

By setting $z = y^{-n+1}$ in the equation, we obtain the linear equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Equation:	$\frac{dy}{dx} + \frac{1}{x}y = xy^2$
Step 1.	$z = y^{-2+1}$
Step 2.	$\frac{dz}{dx} - \frac{1}{x}z = -x$
Step 3.	$u(x) = e^{-\int dx/x} = x^{-1}$
Step 4.	$x^{-1}z = \int -x^{-1}x dx = -\int dx \Rightarrow x^{-1}y^{-1} = -x + C$
Explicit Solution:	$y = (Cx - x^2)^{-1}$

♡ Ricatti's Equation. The nonlinear equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

which frequently occurs in physical applications is called Ricatti's equation. Its solution cannot be expressed in terms of elementary function. However when $R(x) = -1$, we can change the equation into a second order linear differential equation by setting $y = z'/z$.

♡ Clairaut's Equation. The nonlinear equation

$$y = xy' + f(y')$$

is called Clairaut's equation. By differentiating both sides of the equality with respect to x , we obtain the second order equation

$$[x + f'(y')]y'' = 0.$$

One set of solutions called *general solution*, is $y = cx + f(c)$ and is obtained from $y'' = 0$.

If $x + f'(y') = 0$, then we obtain the parametrized curve

$$x = -f'(t), \quad y = f(t) - tf'(t).$$

This curve is also a solution, called the *singular solution*,

♣ Solving an Equation by Inspection. The following identities may help you solve some differential equations.

$yd x + xdy = d(xy)$	$\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right) = -d\left(\frac{y}{x}\right)$
$mx^{m-1}y^n dx + nx^m y^{n-1} dy = d(x^m y^n)$	$\frac{mx^{m-1}y^n dx - nx^m y^{n-1} dy}{y^{2n}} = d\left(\frac{x^m}{y^n}\right)$
$\frac{ydx + xdy}{xy} = d(\ln(xy))$	$\frac{ydx - xdy}{xy} = d\left[\ln\left(\frac{x}{y}\right)\right] = -d\left[\ln\left(\frac{y}{x}\right)\right]$
$\frac{ydx + xdy}{1+x^2y^2} = d[\arctan(xy)]$	$\frac{ydx - xdy}{x^2+y^2} = d\left[\arctan\left(\frac{x}{y}\right)\right] = -d\left[\arctan\left(\frac{y}{x}\right)\right]$

♣ **Picard's Successive Approximations.** Consider the initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

If we integrate both sides of the differential equation from x_0 to x with respect to x , we obtain the new equation

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt. \quad (E_1)$$

Since $y(x_0) = y_0$, the new equation (E_1) is an alternate way of writing the initial-value problem. Furthermore, if we differentiate both sides of (E_1) , we obtain the differential equation

$$y'(x) = f(x, y(x)).$$

We now define a sequence of functions $\{y_n(x)\}$, called *Picard's iterations*, by successive formulas:

Picard's method:	
$y' = f(x, y),$	$y(x_0) = y_0$
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$y_0(x) = y_0$	
$y_1(x) = y_0 + \int_{x_0}^x f[t, y_0(t)] dt,$	
$y_2(x) = y_0 + \int_{x_0}^x f[t, y_1(t)] dt,$	
\vdots	\vdots
$y_n(x) = y_0 + \int_{x_0}^x f[t, y_{n-1}(t)] dt,$	

Equation:	
$y_0 = y(x),$	$y(0) = 1$
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$y_0(x) = y_0 = 1$	
$y_1(x) = 1 + \int_{x_0}^x (1) dt = 1 + x$	
$y_2(x) = 1 + \int_{x_0}^x (1 + t) dt = 1 + x + \frac{x^2}{2},$	
\vdots	\vdots
$y_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$	