The method of successive approximations (or Picard iterations method) could in principle, be used to solve any initial value problem. With this method, we rarely expect to obtain the explicit solution of \( y(x) \) as a function of \( x \). Instead, we construct a table of functions which eventually converge to the exact solution.

**Picard-Lindelf Existence and Uniqueness Theorem.** Consider the Initial Value Problem:

\[
(IVP) \quad \begin{cases} 
  \frac{dy}{dx} = f(x, y) \\
  y(x_0) = y_0. 
\end{cases}
\]

If \( f(x, y) \) and \( \frac{\partial}{\partial y} f(x, y) \) are both continuous on some rectangle \( R \) for which \(-a \leq x \leq a\) and \(-b \leq y \leq b\), that contains the point \((x_0, y_0)\). Then the IVP has a unique solution in some closed interval \( I = [x_0 - h, x_0 + h] \), where \( h > 0 \). Moreover, the Picard iteration

\[
y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t)) \, dt.
\]

produces a sequence of functions \( y_n(x) \) that converges to this solution uniformly on \( I \).

Let \( \Phi_0(x) = y_0 \), then define

\[
\Phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \Phi_0(t)) \, dt.
\]

Clearly this function satisfies the initial condition. Thus \( \Phi_1(x_0) = y_0 \) approximates the unique solution to our differential equation, but likely not well enough.

For a closer approximation to the solution \( y(t) \), we define the following functions, recursively:

\[
\begin{align*}
\Phi_2(x) &= y_0 + \int_{x_0}^{x} f(t, \Phi_1(t)) \, dt \\
\Phi_3(x) &= y_0 + \int_{x_0}^{x} f(t, \Phi_2(t)) \, dt \\
&\vdots = \vdots \\
\Phi_k(x) &= y_0 + \int_{x_0}^{x} f(t, \Phi_{k-1}(t)) \, dt \\
&\vdots = \vdots
\end{align*}
\]

This way

\[
\Phi(x) = \lim_{n \to \infty} \left[ \Phi_0(t) + \int_{x_0}^{x} f(t, \Phi_n(t)) \, dt \right],
\]

is the guaranteed unique solution to the initial value problem and also it is contained in the interval \((h, h) \subseteq (a, a)\).
**Remark 1.** If for some \( k = 0, 1, 2, \ldots \) we have that \( \Phi_k(x) = \Phi_{k+1}(x) \), then \( \Phi_k(x) \) will be the exact solution to our differential equation.

**Remark 2.** If the initial point is not the point \((0,0)\), then we can apply a substitution to translate the differential equation so that \((0,0)\) becomes the initial condition.

Suppose
\[
\begin{align*}
\frac{dy}{dx} &= x(y - x^2 + 2) \\
y(-1) &= 2
\end{align*}
\]
First we choose \( x = t - 1 \) and \( y = z + 2 \). Then by using the chain rule
\[
\frac{dy}{dx} = \frac{d(z + 2)}{d(t - 1)} = \frac{dz}{dt},
\]
we obtain
\[
\begin{align*}
\frac{dz}{dt} &= (t - 1) \left[ (z + 2) - (t - 1)^2 + 2 \right] \\
z(0) &= 0.
\end{align*}
\]

**Example.**
\[
\begin{align*}
\begin{cases}
y' &= x(y - x^2 + 2) \\
y(0) &= 1
\end{cases} & \implies y(x) = 1 + \int_0^x \left[ t(y(t) - t^2 + 2) \right] dt.
\end{align*}
\]

The approximate solutions are:
\[
y_0(x) = y(0) = 1
\]
\[
y_1(x) = 1 + \int_0^x t(y_0(t) - t^2 + 2) dt = 1 + \frac{3}{2} x^2 - \frac{1}{4} x^4
\]
\[
y_2(x) = 1 + \int_0^x t(y_1(t) - t^2 + 2) dt = 1 + \int_0^x \left( 3 + \frac{t^2}{2} - \frac{t^4}{4} \right) dt = 1 + \frac{3}{2} x^2 + \frac{1}{8} x^4 - \frac{1}{24} x^6
\]
\[
y_3(x) = 1 + \int_0^x t(y_2(t) - t^2 + 2) dt = 1 + \frac{3}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 - \frac{1}{192} x^8
\]
\[
\vdots
\]
\[
y_n(x) = x^2 + \left[ 1 + \frac{1}{1!} \frac{x^2}{2} + \frac{1}{2!} \left( \frac{x^2}{2} \right)^2 + \frac{1}{3!} \left( \frac{x^2}{2} \right)^3 + \frac{1}{4!} \left( \frac{x^2}{2} \right)^4 + \cdots \right]
\]
\[
\vdots
\]
\[
y(x) = x^2 + e^{x^2}
\]

Next, we explain how systems of first order nonlinear differential equations with initial conditions may be approximated, using Picard’s iterations method. But first we give an overview of nonlinear systems.

**Nonlinear Systems of Ordinary Differential Equations.** The following system of first order ordinary differential equations:
\[
\begin{align*}
x_1' &= f_1(x_1, x_2, \ldots, x_n) \\
x_2' &= f_2(x_1, x_2, \ldots, x_n) \\
\vdots & \quad \vdots \\
x_n' &= f_n(x_1, x_2, \ldots, x_n)
\end{align*}
\]
is called nonlinear, if there is at least one \( k = 1, 2, \ldots, n \), where the function \( F_k(x_1, x_2, \ldots, x_n) \) is nonlinear (i.e., if one or more variables have degree two or higher and/or there is a product of variables).

One of the greatest difficulties of solving nonlinear problems is that it is not generally possible to combine known solutions into new solutions. In linear problems, for example, a family of linearly independent solutions can be used to construct general solutions through the superposition principle. It is often possible to find several very specific solutions to nonlinear equations, however the lack of a superposition principle prevents the construction of new solutions.

\( \star \) **Jacobian Matrix.** Consider the function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \), where

\[
F(x_1, x_2, \ldots, x_n) = \begin{bmatrix}
f_1(x_1, x_2, \ldots, x_n) \\
f_2(x_1, x_2, \ldots, x_n) \\
\vdots \\
f_m(x_1, x_2, \ldots, x_n)
\end{bmatrix}.
\]

The partial derivatives of \( f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \) (if they exist) can be organized in an \( m \times n \) matrix.

The Jacobian matrix of \( F(x_1, x_2, \ldots, x_n) \) denoted by \( J_F \) is as follows:

\[
J_F(x_1, \ldots, x_n) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}.
\]

Its importance lies in the fact that it represents the best linear approximation to a differentiable function near a given point.

Consider the non-linear multivariable function \( F(t, X(t)) = \mathcal{L}X(t) + \mathcal{N}(X(t)) \), where \( \mathcal{L} \) is composed of linear terms, and \( \mathcal{N} \) is composed of the non-linear terms.

**Example.**

\[
\begin{cases}
x'(t) = x(t) + x^2(t) - x(t)y(t) \\
y'(t) = 2x(t) - 4y(t) + 3x^2(t)y(t)
\end{cases} \iff \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} x^2(t) - x(t)y(t) \\ -3x^2(t)y(t) \end{bmatrix}.
\]

Now, consider the initial value problems of the form

\[
\begin{cases}
X'(t) = F(t, X(t)), \\
X(t_0) = X_0
\end{cases}.
\]

After linearizing locally about a fixed or local state

\[
\mathcal{L} = \frac{\partial F}{\partial X}(X^*); \quad \mathcal{N} = F(X) - \mathcal{L}X.
\]

Here \( \frac{\partial F}{\partial X} \) refers to the partial derivative of \( F \) with respect to \( X \), (the Jacobian of \( F \)).

Exact integration of this problem from time 0 to a later time \( t \) can be performed using matrix exponentials to define an integral equation for the exact solution

\[
X(t) = e^{\mathcal{L}t}X_0 + \int_0^t e^{\mathcal{L}(t-s)}\mathcal{N}(X(s)) \, ds.
\]

This is similar to the exact integral used in the Picard-Lindelf theorem. In the case of \( \mathcal{N} \equiv 0 \), this formulation is the exact solution to the linear differential equation.