

Roots of Polynomials

♣ **Bounds for the Roots of Polynomials.** Let $A = (a_{ij})$ be an $n \times n$ matrix. If $Au = \lambda u$, then λ and u are called the **eigenvalue** and **eigenvector** of A , respectively. The eigenvalues of A are the roots of the **characteristic polynomial**

$$K_A(\lambda) = \det(\lambda I_n - A).$$

The eigenvectors are the solutions to the *Homogeneous system*

$$(\lambda I_n - A)X = \theta.$$

If A is **symmetric**, i.e., $A^t = A$, then all the eigenvalues of A are real. Let

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

be the eigenvalues of A , then

$$\text{trace}(A) = \sum_{k=1}^n \lambda_k = \sum_{k=1}^n a_{kk} \quad \text{and} \quad \det A = \prod_{k=1}^n \lambda_k.$$

Our first theorem is known as the Gerschgorin's Disks Theorem.

Theorem 1. Let $A = (a_{ij})$ be an $n \times n$ matrix. For $j = 1, 2, \dots, n$, define

$$r_j = \left(\sum_{i=1}^n |a_{ij}| \right) - |a_{jj}|.$$

Let $D_j(a_{jj}, r_j)$ be the disk of radius r_j with the center at the point $(0, a_{jj})$ of the complex plane. Then all the eigenvalues of the matrix A are contained within the union of the D_j 's. Thus

$$D(A) = \bigcup_{j=1}^n D_j$$

contains all the eigenvalues of A .

Remark. Since A and A^t have the same set of eigenvalues, we may use Theorem 1. for both A and A^t and get the best neighborhood $D(A) \cap D(A^t)$ for the eigenvalues of A .

Consider now the polynomial of degree n

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n.$$

The polynomial P is said to be **monic**, if the leading coefficient a_0 equals one. Clearly, the matrix

$$Q(x) = \frac{1}{a_0}P(x) = x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n.$$

is monic. To this monic polynomial we associate an $n \times n$ matrix

$$C_P = \begin{pmatrix} -b_1 & -a_2 & b_3 & \dots & -b_{n-1} & -b_n \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The matrix C_P is called the **Companion Matrix** of $P(x)$.

Theorem 2. x_0 is a root of $p(x)$ if and only if x_0 is an eigenvalue of the matrix C_p .

Corollary. Consider a monic polynomial $P(x)$ of degree n . Then

(i) all the roots of $P(x)$ is contained within $D_r \cap D_c$, where

$$D_r = \left[D(0, 1) \cup D \left(-a_{n-1}, \sum_{k=0}^{n-2} |a_k| \right) \right] \quad \text{and}$$

$$D_c = [D(0, |a_{n-1}|) \cup D(0, 1 + |a_{n-2}|) \cup \dots \cup D(0, 1 + |a_2|) \cup D(-a_1, 1)];$$

(ii) if $\{x_1, x_2, \dots, x_n\}$ are the n roots of $P(x)$, then

$$\sum_{k=1}^n x_k = -a_1.$$

Proof. By using C_p , the above theorems, the Remark and the fact that $\text{trace}(C_p) = -a_1$; one may readily prove the corollary.

♣ **Rational Roots.** Although a real polynomial may have complex roots, but there is a well known theorem concerning the rational roots of polynomial with integer coefficients.

Theorem 3. Let $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ be a polynomial with integer coefficients. If p/q is a rational root of $P(x)$, then $a_n = pr$ and $a_0 = qs$.

♣ **Nested Form.** Consider the following polynomial of degree n

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n.$$

The following form of $P(x)$ is called the *nested form* of $P(x)$:

$$P(x) = (((\dots ((a_0)x + a_1)x + a_2)x \dots)x + a_{n-1})x + a_n).$$

Finally, we present a root finding tool known as *Horner's method* or *Synthetic division*.

♣ **Synthetic Division.** Consider the polynomial:

$$P(x) = 2x^4 - 3x^2 + 3x - 4 = (((((2)x + 0)x - 3)x + 3)x - 4) = -4 + x(3 + x(-3 + x(0 + x(2))))).$$

The following chart shows how to evaluate $P(a), P'(a), P''(a), \dots$ for $a = -2$.

$-2 \mid$	$\searrow \searrow \searrow$ \times	2	0	-3	3	-4	
		$\updownarrow +$	$\updownarrow +$	$\updownarrow +$	$\updownarrow +$	$\updownarrow +$	
		0	-4	8	-10	14	
		2	-4	5	-7	<u>10 = 10 × 0! = p(-2)</u>	
		0	-4	16	-42		
		2	-8	21	-49	<u>-49 × 1! = -49 = p'(-2)</u>	
		0	-4	24			
		2	-12	45		<u>45 × 2! = 90 = p''(-2)</u>	