Roots of Polynomials

♣ **Bounds for the Roots of Polynomials**. Let $A = (a_{ij})$ be an $n \times n$ matrix. If $Au = \lambda u$, then λ and u are called the **eigenvalue** and **eigenvector** of A, respectively. The eigenvalues of A are the roots of the **characteristic polynomial**

$$K_A(\lambda) = \det(\lambda I_n - A).$$

The eigenvectors are the solutions to the *Homogeneous system*

$$(\lambda I_n - A) X = \theta.$$

If A is **symmetric,** i.e., $A^t = A$, then all the eigenvalues of A are real. Let

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

be the eigenvalues of A, then

$$trace\,(A) = \sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} a_{ii} \qquad and \qquad \det A = \prod_{k=1}^{n} \lambda_k\,.$$

Our first theorem is known as the Gerschgorin's Disks Theorem.

Theorem 1. *Let $A = (a_{ij}$ be an $n \times n$ matrix. For $j = 1, 2, \ldots, n$, define*

$$r_j = \left(\sum_{i=1}^{n} |a_{ij}| \right) - |a_{ij}|.$$

Let $D_j\,(a_{jj}, r_j)$ be the disk of radius r_j with the center at the point $(0, a_{jj})$ of the complex plane. Then all the eigenvalues of the matrix A are contained within the union of the D_j's. Thus

$$D(A) = \bigcup_{j=1}^{n} D_j$$

contains all the eigenvalues of A.

Remark. Since A and A^t have the same set of eigenvalues, we may use Theorem 1. for both A and A^t and get the best neighborhood $D(A) \cap D(A^t)$ for the eigenvalues of A.

Consider now the polynomial of degree n

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n.$$

The polynomial P is said to be **monic**, if the leading coefficient a_0 equals one. Clearly, the matrix

$$Q(x) = \frac{1}{a_0}P(x) = x^n + b_1 x^{n-1} + \ldots + b_{n-1}x + b_n.$$

is monic. To this monic polynomial we associate an $n \times n$ matrix

$$C_P = \begin{pmatrix} -b_1 & -a_2 & b_3 & \ldots & -b_{n-1} & -b_n \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

The matrix C_P is called the **Companion Matrix** of $P(x)$.

Theorem 2. *x_0 is a root of $p(x)$ if and only if x_0 is an eigenvalue of the matrix C_p.*

Corollary. *Consider a monic polynomial $P(x)$ of degree n. Then*

(i) all the roots of $P(x)$ is contained within $D_r \cap D_c$, where

$$D_r = \left[D(0,1) \bigcup D\left(-a_{n-1}, \sum_{k=0}^{n-2} |a_k| \right) \right] \qquad \textit{and}$$

$$D_c = [D(0,|a_{n-1}|) \cup D(0, 1+|a_{n-2}|) \cup \ldots \cup D(0, 1+|a_2|) \cup D(-a_1, 1)]\,;$$

(ii) if $\{x_1, x_2, \ldots, x_n\}$ are the n roots of $P(x)$, then

$$\sum_{k=1}^{n} = -a_1.$$

Proof. By using C_p, the above theorems, the Remark and the fact that $trace(C_p) = -a_1$; one may readily prove the corollary.

♣ **Rational Roots**. Although a real polynomial may have complex roots, but there is a well known theorem concerning the rational roots of polynomial with integer coefficients.

Theorem 3. *Let $P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n$ be a polynomial with integer coefficients. If p/q is a rational root of $P(x)$, then $a_n = pr$ and $a_0 = qs$.*

♣ **Nested Form**. Consider the following polynomial of degree n

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n.$$

The following form of $P(x)$ is called the *nested form* of $P(x)$:

$$P(x) = (((\ \cdots \ (((a_0)x + a_1)x + a_2)x \ \cdots \ )x + a_{n-1})x + a_n).$$

Finally, we present a root finding tool known as *Horner's method* or *Synthetic division*.

♣ **Synthetic Division**. Consider the polynomial:

$$P(x) = 2x^4 - 3x^2 + 3x - 4 = (((((2)x + 0)x - 3)x + 3)x - 4) = -4 + x(3 + x(-3 + x(0 + x(2)))).$$

The following chart shows how to evaluate $P(a), P'(a), P''(a), ...$ for $a = -2$.

-2 |	2	0	-3	3	-4
	$\updownarrow +$	$\updownarrow +$	$\updownarrow +$	$\updownarrow +$	$\updownarrow +$
$\searrow\searrow\searrow$ $\times$	0	-4	8	-10	14
	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	
	2	-4	5	-7	$10 = 10 \times 0! = p(-2)$
	0	-4	16	-42	
	2	-8	21	-49	$-49 \times 1! = -49 = p'(-2)$
	0	-4	24		
	2	-12	45		$45 \times 2! = 90 = p''(-2)$