**LPP1:** Consider the *Linear Programming Problem:*

Maximize \( z = 30x_1 + 6x_2 - 5x_3 + 18x_4 \)

subject to:

\[
\begin{align*}
    x_1 + 2x_3 + x_4 & \leq 20 \\
    -2x_1 + x_2 - x_4 & \leq 15 \\
    6x_1 + 2x_2 - 3x_3 & \leq 54
\end{align*}
\]

where \( x_1, x_2, x_3, x_4 \geq 0 \).

The initial and final tableaux of the LPP1 are as follows:

<table>
<thead>
<tr>
<th></th>
<th>( s_5 )</th>
<th>( s_6 )</th>
<th>( s_7 )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>-2</td>
<td>0</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>( s_6 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_7 )</td>
<td>5</td>
<td>18</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( s_5 )</th>
<th>( s_6 )</th>
<th>( s_7 )</th>
<th>( b^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>4</td>
<td>0</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>( s_6 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s_7 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>27</td>
</tr>
</tbody>
</table>

**Initial Tableau**

**Final Tableau**

Notice that in the final tableau, the identity matrix \( I_3 = [A^*_5, A^*_6, A^*_7] \).

In the initial tableau, we use \( A_4, A_6 \) and \( A_2 \) to create the matrix

\( B = [A_4, A_6, A_2] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \).

Since \( B^{-1}I_3 = B^{-1} \), we conclude that the inverse of \( B \) is obtain from \( A^*_5, A^*_6, A^*_7 \) in the final tableau. Thus

\( B^{-1} = [A^*_5, A^*_6, A^*_7] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \).

Note that \( B^{-1}A = A^* \), so \( B^{-1}A_j = A^*_j \), and also \( B^{-1}b = b^* \).

From the vector \( c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (-30, -6, 5, -18, 0, 0, 0) \), we define the vector

\( c_B = (c_4, c_6, c_2) = (-18, 0, -6) \)

\( c^* = c - c_BB^{-1}A = c - c_B A^* \).

\( z^* = z - c_BB^{-1}b = z - c_B b^* \)

Hence the final tableau is

\( A^* = B^{-1}A \quad \quad b^* = B^{-1}b \)

\( c^* = c - c_B B^{-1}A \quad z^* = z - c_B B^{-1}b \).

**Remark 1.** In any maximization problem in standard form, if the final tableau is given, then the inverse of the matrix \( B \) is just the columns under the slack variables in the final tableau.

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The Dual of (LPP1) is as follows:

Minimize $v = 20y_1 + 15y_2 + 54y_3$

subject to:

\[
\begin{align*}
& y_1 - 2y_2 + 6y_3 \geq 30 \\
& \quad + y_2 + 2y_3 \geq 6 \\
& 2y_1 - 3y_3 \geq -5 \\
& y_1 - y_2 \geq 18
\end{align*}
\]

where $y_1, y_2, y_3 \geq 0$.

The solution to this dual problem may be obtained from the final tableau of the Primal problem:

\[(y_1^*, y_2^*, y_3^*) = (c_5, c_6, c_7) = (18, 0, 3)\].

### Changes in the Objective Function

The changes in the objective vector $c$ can induce changes only in $c^*$ and $z^*$, and in fact the new $c^*$ can be determined using $c^* = c - c_B B^{-1} A$. If the modified $c^*$ remains non-negative, $X^*$ would remain an optimal solution; if not, more iterations of the simplex algorithm may be necessary to complete the modified problem.

(i) Suppose the coefficients of $x_1$ and $x_3$ in our LPP1 fluctuate. Since $x_1^* = 6$ and $x_3^* = 32$ are not basic variables in the optimal solution of our final tableau, then in $c_1^* = (-30 - \lambda_1, -6, 5 - \lambda_3, -18, 0, 0, 0)$, as long as $6 - \lambda_1 \geq 0$ and $32 - \lambda_3 \geq 0$ (i.e., $\lambda_1 \leq 6$ and $\lambda_3 \leq 32$), $X^*$ would remain optimal. Otherwise, we must use more iterations of the simplex algorithm on the optimal tableau, in order to make all the components of $c^*$ non-negative.

(ii) Suppose the coefficients of $x_2$ or $x_4$ in our LPP1 fluctuate. Since $x_2^*$ and $x_4^*$ are both basic variables in the optimal solution of our final tableau, then in $c_2^* = (-30, -6 - \lambda_2, 5, -18, 0, 0, 0)$ or $c_4^* = (-30, -6, 5, -18 - \lambda_4, 0, 0, 0)$, we must find the lower and upper bounds of $\lambda_2$ or $\lambda_4$ which keep the components of $c_{\lambda_2}^*$ or $c_{\lambda_4}^*$ non-negative.

To do this; for $\lambda_2$, we first find:

\[
c_{\lambda_2}^* = c^* - (0, 0, -\lambda_2) [A^* | b^*] = \left[6 + 3\lambda_2, \lambda_2, 32 - \frac{3}{2}\lambda_2, 0, 18, 0, 3 + \frac{\lambda_2}{2} | 522 + 27 \lambda_2 \right].
\]

Then, we need to solve the system of inequalities:

\[
\begin{align*}
6 + 3\lambda_2 & \geq 0 \\
\lambda_2 & \geq 0 \\
32 - \frac{3}{2}\lambda_2 & \geq 0 \\
3 + \frac{\lambda_2}{2} & \geq 0
\end{align*}
\]

To find the lower and upper bounds of $\lambda_4$ which keeps $X^*$ optimal, first we find $c_{\lambda_4}^* = c^* - (-\lambda_4, 0, 0) [A^* | b^*] = [6 + \lambda_4, 0, 32 + 2\lambda_4, \lambda_4, 18 + \lambda_4, 0, 3 | 522 + 20\lambda_4]$.  

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Then, we need to solve the system of inequalities:

\[
\begin{cases}
6 + \lambda_4 \geq 0 \\
32 + 2\lambda_4 \geq 0 \\
\lambda_4 \geq 0 \\
18 + \lambda_4 \geq 0
\end{cases} \implies \lambda_4 \geq 0.
\]

**Changes in the Constant Column Vector**

Changing the original column vector \(b\) into \(b'\) will affect \(b^*\) and \(z^*\) of the final tableau, but not \(c^*\) and \(A^*\).

a. If the entries of \(b^* = B^{-1}b'\) remain non-negative, then since \(c^* \geq 0\), the optimal solution to the modified problem will have the same optimal solution as the original problem, with values given by \(b^*; \) and \(z^* = z - c_B b^*\).

b. If some entries of \(b^*\) are negative, then to resolve this problem we use the Dual Simplex Algorithm on this new tableau.

**LPP2:** Consider the Linear Programming Problem:

Maximize \(z = 5x_1 + 3x_2\)

subject to:

\[
\begin{pmatrix}
4 & 2 \\
4 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\leq
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}
\leq
\begin{pmatrix}
13 \\
10 \\
5
\end{pmatrix}
\]

where \(b_i \in \mathbb{N} \quad i = 1, 2, 3, \quad \text{and} \quad x_j \geq 0 \quad j = 1, 2.\)

(i) Find a vector \(b = (b_1, b_2, b_3)^t, \quad b_i \in \mathbb{N}, \) for which \((2, 2)^t\) is an optimal solution.

The canonical form of this LP problem produces the following matrix and vectors:

\[
A = \begin{pmatrix}
4 & 2 & 1 & 0 & 0 \\
4 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
13 \\
10 \\
5
\end{pmatrix}, \quad \text{and} \quad c = (-5, -3, 0, 0, 0).
\]

Note that \(A \begin{pmatrix}2 \\ 2 \end{pmatrix} = \begin{pmatrix}10 \\ 10 \end{pmatrix} \leq \begin{pmatrix}13 \\ 10 \end{pmatrix}, \) which makes \(\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}2 \\ 2 \end{pmatrix}\) a feasible solution.

Now we need to choose the first two columns and one of the other columns of \(A\), to define the matrix \(B\). Since finding the inverse of a \(3 \times 3\) block triangular matrix is easier than of a non-block triangular matrix, we move the first rows of the matrix \(A\) and the vector \(b\) to the third position. This way, we obtain:

\[
A' = \begin{pmatrix}
4 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
4 & 2 & 1 & 0 & 0
\end{pmatrix}, \quad b' = \begin{pmatrix}
10 \\
5 \\
13
\end{pmatrix}, \quad B' = \begin{pmatrix}
4 & 1 & 0 \\
1 & 1 & 0 \\
4 & 2 & 1
\end{pmatrix}, \quad \text{with} \quad B'^{-1} = \frac{1}{3} \begin{pmatrix}
1 & -1 & 0 \\
-1 & 4 & 0 \\
-2 & -4 & 3
\end{pmatrix}.
\]

Now we have:

\[
A'^* = B'^{-1}A' = \begin{pmatrix}
1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\
0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \\
0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3}
\end{pmatrix}, \quad b'^* = B'^{-1} \begin{pmatrix}
10 + \lambda_1 \\
5 + \lambda_2 \\
13 + \lambda_3
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
5 + \lambda_1 - \lambda_2 \\
10 - \lambda_1 + 4 \lambda_2 \\
-1 - 2\lambda_1 - 4 \lambda_2 + 3 \lambda_3
\end{pmatrix} = \begin{pmatrix}
2 \\
\lambda
\end{pmatrix}.
\]
Sensitivity Analysis

We have \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \). As long as \( b^*_3 = 1 + \lambda_3 \geq 0 \), the vector \( \left( \frac{2}{2} \right) \) will remain optimal; so we may choose for example \( \lambda_3 = -1 \). Thus for the vector
\[
\left( \begin{array}{c}
 b^*_1 \\
 b^*_2 \\
 b^*_3 
\end{array} \right) = \left( \begin{array}{c}
 13 - 1 \\
 10 - 0 \\
 5 - 1 
\end{array} \right) = \left( \begin{array}{c}
 12 \\
 10 \\
 4 
\end{array} \right),
\]
the vector \( \left( \frac{2}{2} \right) \) is an optimal solution.

**Remark 2.** Note that the fourth or the fifth column of \( A \) could have been used to define the matrix \( B \). By choosing the fifth column of \( A \), the matrix \( B \) would be a block triangular matrix; so there is no need for moving some rows. Clearly other results may have been produced. Also other values of \( \lambda_3 \geq -1 \) may have been used.

(ii) Find the solution to the corresponding dual problem of the LP problem that you created, in which \( (2, 2)^t \) was an optimal solution.

The Dual of the problem is:

Minimize \( v = 10y_1 + 4y_2 + 12y_3 \)

subject to:
\[
\left( \begin{array}{ccc}
 4 & 1 & 4 \\
 1 & 1 & 2 
\end{array} \right) \left( \begin{array}{c}
 y_1 \\
 y_2 \\
 y_3 
\end{array} \right) \geq \left( \begin{array}{c}
 5 \\
 3 
\end{array} \right) \text{ with } y_1, y_2, y_3 \geq 0.
\]

To find the solution to this problem, we only need to find \( c^* \) and \( z^* \) of part (a).

\[
[c^* \ z^*] = [c \mid 0] - c_B [A^* \mid b^*] = [-5, -3, 0, 0, 0 \mid 0] - (-5, -3, 0, 0, 0 \mid 0) = \left[ \begin{array}{c}
 0 \\
 0 \\
 2 \frac{2}{3} \\
 7 \frac{7}{3} \\
 -16 
\end{array} \right].
\]

Hence \( y_1 = 0 \), \( y_2 = \frac{2}{3} \), and \( y_3 = \frac{7}{3} \) with the maximum of \( v \) equal to 16.

**Remark 3.** Note that this LP problem is not exactly the dual of the original LPP2; since for the convenience of finding the inverse of a matrix \( B \), we moved the rows of the original matrix \( A \).

**LPP3** A small furniture company produces chairs, desks, and tables. The manufacture of each type of furniture requires lumber and carpentry labor. The profit from each item, the material and labor requirements, and the amount of available lumber and labor are given in the table.

<table>
<thead>
<tr>
<th>Chair</th>
<th>Desk</th>
<th>Table</th>
<th>Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lumber (board feet)</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Labor (hours)</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Profit (Dollars)</td>
<td>40</td>
<td>110</td>
<td>20</td>
</tr>
</tbody>
</table>

The initial and final tableaus for this LP problem is as follows:

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_4 )</th>
<th>( s_5 )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_4 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( c )</td>
<td>-40</td>
<td>-110</td>
<td>-20</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_1^* )</th>
<th>( x_2^* )</th>
<th>( x_3 )</th>
<th>( s_4 )</th>
<th>( s_5 )</th>
<th>( b^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1^* )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>( x_2^* )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( c^* )</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>10</td>
<td>30</td>
</tr>
</tbody>
</table>
From the final tableau, we obtain

\[ B^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}, \quad b^* = \begin{pmatrix} 10 \\ 40 \end{pmatrix}, \quad c^* = (0, 0, 50, 10, 30), \quad c_B = (-40, -110), \quad \text{and} \quad z^* = 4800. \]

(i) How large can the profit from the chairs be before a change in the optimal solution is required?

\[ c_{\lambda_1}^* = c^* - (-\lambda_1, 0) [A^* | b^*] = (\lambda_1, 0, 50 - \lambda_1, 10 + 3\lambda_1, 30 - 2\lambda_1). \]

Thus \( 0 \leq \lambda_1 \leq 15 \). This means that by charging less than $40 or more than $55 for each chair, the final tableau must be changed. At $55 per chair, the maximum profit will be $4950.

(ii) If in the original problem the availability of lumber decreases from 90 to 80 board feet and (at the same time) the availability of labor increases from 130 to 160 hours, what is the new optimal solution?

We have

\[ B^{-1} \begin{pmatrix} 80 \\ 160 \end{pmatrix} = \begin{pmatrix} -80 \\ 80 \end{pmatrix} \quad \text{and} \quad c_B \begin{pmatrix} -80 \\ 80 \end{pmatrix} = 5600. \]

By changing the third column of our final tableau, we use the dual simplex algorithm to obtain our final tableau.

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_4 )</th>
<th>( s_5 )</th>
<th>( b^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1^* )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>( x_2^* )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( c^* )</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>( s_5^* )</td>
<td>-( \frac{1}{2} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( x_2^* )</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( c^* )</td>
<td>15</td>
<td>0</td>
<td>35</td>
<td>55</td>
<td>0</td>
</tr>
</tbody>
</table>

**Addition of a New Variable**

Suppose that now we wish to add another variable in the formulation of the original problem. Let \( x_{n+1} \) be the new variable, with the objective coefficient \( c_{n+1} \) and the column vector of coefficients for the constraining equations \( A_{n+1} \). Then the expanded, modified problem in the canonical form is to

Minimize \( z' = c'X' \),

subject to: \( A'X' = b' \),

\( X' \geq 0 \),

Now \( X^* \) is still a basic feasible solution to the new problem if we simply set the value of the non-basic variable \( x_{n+1} \) equal to 0. Moreover, this point will provide an optimal solution if \( c_{n+1}^* = c_{n+1} - c_B B^{-1} A_{n+1} \geq 0 \); if not, we just use the simplex algorithm on the new tableau by using the \( n + 1^{th} \) column as a pivot column.

**Remark 4.** We may use the same technique, if the component of one of the variables and its coefficient in the objective function is altered.
(iii) Consider again the original LPP3.

Suppose there is a proposal to replace the current model of table by a new model that requires 3 board feet of lumber and 2 hours of labor. If the profit from this model of table is expected to be 100, should this new model be produced?

The new initial tableau is now

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_4$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>$s_5$</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>130</td>
</tr>
<tr>
<td>$c$</td>
<td>-40</td>
<td>-110</td>
<td>-100</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We only need to multiply the matrix $B^{-1}$ by the third column of the new matrix $A$ and deal only with the third component of $c$.

$$B^{-1} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad \text{and} \quad 100 - (-40, -110) \begin{pmatrix} 5 \\ -1 \end{pmatrix} = -100 + 90 = -10.$$ 

Now we need to use the simplex algorithm to get the optimal solution.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$b^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^*$</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>-2</td>
<td>10</td>
</tr>
<tr>
<td>$x_2^*$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>$c^*$</td>
<td>0</td>
<td>0</td>
<td>-10</td>
<td>10</td>
<td>30</td>
<td>4800</td>
</tr>
<tr>
<td>$x_3^*$</td>
<td>$\frac{1}{5}$</td>
<td>0</td>
<td>1</td>
<td>$\frac{3}{5}$</td>
<td>$-\frac{2}{5}$</td>
<td>2</td>
</tr>
<tr>
<td>$x_2^*$</td>
<td>$\frac{1}{5}$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{2}{5}$</td>
<td>$\frac{3}{5}$</td>
<td>42</td>
</tr>
<tr>
<td>$c^*$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>16</td>
<td>26</td>
<td>4820</td>
</tr>
</tbody>
</table>

**Addition of a Constraint**

Suppose after solving the problem we wish to alter the original problem by the addition of a new constraint. Now it could be that $X^*$ satisfies this new constraint. If this is the case, $X^*$ is also optimal for the expanded problem, because clearly, by this addition of a constraint, we have not changed the objective function nor increased the set of feasible solutions to the system of constraints. On the other hand, if $X^*$ does not satisfy this new constraint, we must find a new optimal solution. Under certain circumstances, however, this problem may be resolved quite easily by creating a new canonical tableau. If the constant column of the new canonical tableau contains some negative entries, then by using the dual simplex algorithm, we may solve the problem.

(iv) Consider once again the original LPP3.

Suppose it is required that the number of chairs be at least as great as the number of desks. What is the optimal solution in that case?
The fact that the optimal solution \((x_1^*, x_2^*, x_3^*) = (10, 40, 0)\) doesn’t satisfy this condition, we need to add a new constraint.

\[x_2 \leq x_1 \quad \Rightarrow \quad -x_1 + x_2 \leq 0.\]

Using the final tableau of the original problem and adding a new row and a new slack variable, we try to make \(x_1\) and \(x_2\) basic variables; and use the dual simplex method if necessary to find the optimal solution.

\[
\begin{array}{ccccccc}
 x_1 & x_2 & x_3 & s_4 & s_5 & s_6 & b \\
 \hline
 x_1 & 1 & 0 & -1 & 3 & -2 & 0 & 10 \\
 x_2 & 0 & 1 & 1 & -1 & 1 & 0 & 40 \\
 s_6 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\
 \hline
 c & 0 & 0 & 50 & 10 & 30 & 0 & 4800 \\
 \hline
 x_1 & 1 & 0 & -1 & 3 & -2 & 0 & 10 \\
 x_2 & 0 & 1 & 1 & -1 & 1 & 0 & 40 \\
 s_6 & 0 & 0 & -2 & 4 & -3 & 1 & -30 \\
 \hline
 c & 0 & 0 & 50 & 10 & 30 & 0 & 4800 \\
 \hline
 x_1^* & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & -\frac{2}{3} & 30 \\
 x_2^* & 0 & 1 & -\frac{4}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 30 \\
 s_5^* & 0 & 0 & \frac{2}{3} & -\frac{4}{3} & 1 & -\frac{1}{3} & 10 \\
 \hline
 c^* & 0 & 0 & 30 & 50 & 0 & 10 & 4500 \\
\end{array}
\]