

Series Solutions of Linear Differential Equations

In this chapter we shall solve some second-order linear differential equation about an initial point using The *Taylor series*.

In all that follows we assume that (E) is a second-order homogeneous linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad (1)$$

where the coefficients $a_0(x)$, $a_1(x)$, and $a_2(x)$ are polynomials. The *normalized form* of (E),

$$y'' + P(x)y' + Q(x)y = 0. \quad (2)$$

is obtained when the equation is divided by the polynomial $a_0(x)$.

A function $f(x)$ is said to be *analytic* at x_0 if its *Taylor series* about x_0 ,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k,$$

exists and converges to $f(x)$ for all x in some open interval containing x_0 . Functions such as polynomials, e^x , $\sin x$, and $\cos x$ are analytic at all points. A rational function is analytic at all points except, when the denominator becomes zero.

If both $P(x)$ and $Q(x)$ are analytic at x_0 , then x_0 is called an *ordinary* point of the equation. If either (or both) of these function is not analytic at x_0 , then x_0 is called a *singular* point of the differential equation.

Let x_0 be a singular point of (E), then we define $deg_P(x_0)$ (rep $deg_Q(x_0)$) as the power of $(x - x_0)$ in the denominator of $P(x)$ (resp. $Q(x)$). If

$$0 \leq deg_P(x_0) \leq 1 \quad 0 \leq deg_Q(x_0) \leq 2,$$

then x_0 is called a *regular singular point* (RSP), otherwise x_0 is said to be *irregular singular point* (ISP).

Consider the differential equation

$$x^3(x+1)^2(x-2)^3(x^2+1)y'' + x^2(x+1)(x^2-4)^2y' + xy = 0.$$

The normalized form of this equation is

$$y'' + \frac{(x+2)^2}{x(x+1)^2(x-2)(x^2+1)}y' + \frac{1}{x^2(x+1)^2(x-2)^3(x^2+1)}y = 0.$$

The point $0, -1, 2, -i,$ and i are singular points. All singular points, except 2 are regular.

♣ **Power Series Solutions About an Ordinary Point.** If the point x_0 is an ordinary point of the differential equation (E), then there is a power series solutions of the form

$$\sum_{k=0}^{\infty} C_k(x - x_0)^k \quad (3)$$

for the equation (E).

Now that we are assured that the equation about the regular point has a power series solution; we need to find the coefficients C_k 's in (3).

Algorithm. We assume that

$$y = \sum_{k=0}^{\infty} C_k(x - x_0)^k.$$

Hence

$$y' = \sum_{k=1}^{\infty} kC_k(x - x_0)^{k-1} \quad \text{and} \quad y'' = \sum_{k=2}^{\infty} k(k-1)C_k(x - x_0)^{k-2}.$$

Next we put the power series corresponding to $y, y',$ and y'' in (2) (the non-normalized form of the equation (E)), we obtain

$$a_0(x) \left[\sum_{k=2}^{\infty} k(k-1)C_k(x - x_0)^{k-2} \right] + a_1(x) \left[\sum_{k=1}^{\infty} kC_k(x - x_0)^{k-1} \right] + a_2(x) \left[\sum_{k=0}^{\infty} C_k(x - x_0)^k = 0 \right].$$

We now need to set the coefficient of each $(x - x_0)$ term to zero. We illustrate the whole procedure with the following example.

Example. Consider the differential equation

$$y'' + xy' + (x^2 + 2)y = 0. \quad (4)$$

Notice that the point $x_0 = 0$ is an ordinary point, so we assume that

$$y = \sum_{k=0}^{\infty} C_k x^k$$

is a power series solution to the equation (4). Differentiating term by term we obtain

$$y' = \sum_{k=1}^{\infty} kC_k x^{k-1} \quad \text{and} \quad y'' = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}.$$

Substituting the series for $y, y',$ and y'' in (4), we obtain

$$\left[\sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} \right] + x \left[\sum_{k=1}^{\infty} kC_k x^{k-1} \right] + (x^2 + 2) \left[\sum_{k=0}^{\infty} C_k x^k = 0 \right] = 0. \quad (5)$$

Multiplying x by y' and $(x^2 + 1)$ by y in (5), we obtain

$$\left[\sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} \right] + \left[\sum_{k=1}^{\infty} kC_k x^k \right] + \left[\sum_{k=0}^{\infty} C_k x^{k+2} \right] + 2 \left[\sum_{k=0}^{\infty} C_k x^k \right] = 0. \quad (6)$$

In order to match the powers of x 's in (6), we need to start k in each series from different values. Thus (6) becomes

$$\left[\sum_{k=0}^{\infty} (k+2)(k+1)C_{k+2} x^k \right] + \left[\sum_{k=1}^{\infty} kC_k x^k \right] + \left[\sum_{k=2}^{\infty} C_{k-2} x^k \right] + 2 \left[\sum_{k=0}^{\infty} C_k x^k \right] = 0. \quad (7)$$

Now we start k from 2 in all four power series in (7); so we need to extract terms with indices less than two. Once this is done we combine all of the power series into a unique power series. The equation (7) then becomes

$$[2C_2 + 6C_3x] + [C_1x] + [2C_0 + 2C_1x] + \left[\sum_{k=0}^{\infty} [(k+2)(k+1)C_{k+2} + kC_k + C_{k-2} + C_k] x^k \right] = 0. \quad (8)$$

From (8) we obtain the system

$$\begin{cases} 2C_2 + 2C_0 = 0 \\ 6C_3 + 3C_1 = 0 \end{cases} \quad (9)$$

and a formula

$$[(k+2)(k+1)C_{k+2} + kC_k + C_{k-2} + C_k] = 0 \quad (10)$$

called the *recurrence formula*.

The system (9) gives us

$$C_2 = -C_0 \quad \text{and} \quad C_3 = -\frac{1}{2}C_1. \quad (11)$$

The conditions (11) and the recurrence formula enable us to express each coefficient C_{k+2} for $k \geq 2$ in terms of the previous coefficients C_k and C_{k-2} , thus giving

$$C_{k+2} = -\frac{(k+1)C_k + C_{k-2}}{(k+1)(k+2)}, \quad (k \geq 2) \quad (12)$$

♣ Power Series Solutions About a Regular Singular Point. We shall restrict our study to the interval $x > 0$ and if we then wish to find solutions for negative interval, by substituting $u = -x$ in (E), we may study the resulting equation for positive u . If the singular point is not zero, then by translating the origin to that point; this way, we obtain a new equation with a singular point at $x_0 = 0$. Since the equation (E) behave badly at an irregular singular point, we only study the case of regular singular points.

♡ **The Method of Frobenius.** If x_0 is a regular singular point of the differential equation (E), then for some real or complex constant (which may be determined), the power series

$$\sum_{k=0}^{\infty} C_k x^{k+r}, \quad (C_0 \neq 0) \quad (12)$$

is a solution to (E).

The fact that $x_0 = 0$ is a RSP implies that the Mac Laurin series of

$$xP(x) = p_0 + xp_1 + x^2p_2 + \dots = \sum_{k=0}^{\infty} p_k x^k \quad \text{and}$$

$$x^2Q(x) = q_0 + xq_1 + x^2q_2 + \dots = \sum_{k=0}^{\infty} q_k x^k$$

By putting the power series of y , $xP(x)$ and $x^2Q(x)$ into the differential equation (E), and equating the lowest term of this series to zero, we obtain the equation

$$r^2 + (p_0 - 1)r + q_0 = 0. \quad (13)$$

This equation is called the *indicial equation of (E)*.

Let r_1 and r_2 be the roots of the indicial equation with $Re(r_1) \geq Re(r_2)$.

Case 1. If $r_1 - r_2$ is not an integer, then the linearly independent solutions of the equation (E) are given respectively by

$$y_1(x) = x_1^r \sum_{k=0}^{\infty} C_k^1 x^k$$

where $C_0 \neq 0$, and

$$y_2(x) = x_2^r \sum_{k=0}^{\infty} C_k^2 x^k$$

where $C_0 \neq 0$.

Case 2. If $r_1 - r_2$ is an integer, then the linearly independent solutions of the equation (E) are given respectively by

$$y_1(x) = x_1^r \sum_{k=0}^{\infty} C_k^1 x^k$$

where $C_0^1 \neq 0$, and

$$y_2(x) = x_2^r \sum_{k=0}^{\infty} C_k^2 x^k + C y_1(x) \ln x$$

where $C_0^2 \neq 0$, and C is a constant which may or may not be zero.

Case 3. If $r_1 = r_2$, the the linearly independent solutions of the equation (E) are given respectively by

$$y_1(x) = x_1^r \sum_{k=0}^{\infty} C_k^1 x^k$$

where $C_0^1 \neq 0$, and

$$y_2(x) = x_2^r \sum_{k=0}^{\infty} C_k^2 x^k + y_1(x) \ln x$$

where $C_0^2 \neq 0$.

Equation:	$2x^2y'' + xy' + (x^2 - 3)y = 0$
Indicial Equation	$2r^2 - r - 3 = 0$
Indicial Roots	$r_1 = \frac{3}{2} \quad r_2 = -1$
Recurrence formulas	$\begin{cases} r_1 = \frac{3}{2} & \{ C_1 = 0 \quad C_k = -\frac{C_{k-2}}{n(2n+5)} \quad n \geq 2, \\ r_2 = -1 & \{ C_1 = 0 \quad C_k = -\frac{C_{k-2}}{n(2n-5)} \quad n \geq 2. \end{cases}$

Equation:	$x^2y'' - xy' - (x^2 - \frac{5}{4})y = 0$
Indicial Equation	$r^2 - 2r - \frac{5}{4} = 0$
Indicial Roots	$r_1 = \frac{5}{2} \quad r_2 = -\frac{1}{2}$
Recurrence formulas	$\begin{cases} r_1 = \frac{5}{2} & \{ C_1 = 0 \quad C_k = \frac{C_{k-2}}{n(n+3)} \quad n \geq 2, \\ r_2 = -1 & \{ C_1 = 0 \quad C_k = \frac{C_{k-2}}{n(n-3)} \quad n \geq 2. \end{cases}$

♠ **Note.** If $r = \alpha \pm \beta i$ are the complex roots of the indicial equation, then the linearly independent solution to the differential equation (E) are

$$\begin{aligned} y(x) &= x^r \sum_{k=0}^{\infty} C_k x^k = x^{\alpha \pm \beta i} \sum_{k=0}^{\infty} C_k x^k = x^{\alpha} e^{\pm i\beta \ln x} \sum_{k=0}^{\infty} C_k x^k \\ &= x^{\alpha} [\cos(\beta \ln x) \pm \sin(\beta \ln x)] \sum_{k=0}^{\infty} C_k x^k \\ &= x^{\alpha} \cos(\beta \ln x) \sum_{k=0}^{\infty} C_k x^k \pm \sin(\beta \ln x) \sum_{k=0}^{\infty} C_k x^k. \end{aligned}$$