An integral transform maps an equation from its original domain into another domain, where it might be manipulated and solved much more easily than in the original domain. The solution is then mapped back to the original domain using the inverse of the integral transform. Many functions in analysis are defined and expressed as improper integrals of the form

\[ F(u) = \int_{t_1}^{t_2} f(t) K(t, u) \, dt. \]

A function \( F(u) \) defined in this way (\( t \) may be either a real or a complex variable) is called an integral transform of \( f(t) \). The function \( K(t, u) \) which appears in the integrand is referred to as the kernel of the transform.

There are numerous useful integral transforms. Each is specified by the kernel \( K(t, u) \).

Some kernels have an associated inverse kernel \( K^{-1}(t, u) \) which (roughly speaking) yields an inverse transform:

\[ f(t) = \int_{u_1}^{u_2} f(u) K^{-1}(t, u) \, du. \]

Integral transforms are used very extensively in both pure and applied mathematics, as well as in science and engineering. They are especially useful in solving certain boundary value problems, partial differential equations, and some types of integral equations.

An integral transform is a linear and invertible transformation, and a partial differential equation can be reduced to a system of algebraic equations by application of an integral transform. The algebraic problem is easy to solve for the transformed function \( F \) and the function \( f \) can be recovered from \( F \) by some inversion formula.

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Solution to the PDE

Some of the more commonly used integral transforms are the following:

1. The Fourier Transform (FT):
   \[ F(\omega) = [\mathcal{F} f(t)](\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t) \, dt. \]

2. The Fourier Cosine Transform (FCT):
   \[ F_c(\omega) = [\mathcal{F}_c f(t)](\omega) = \int_{-\infty}^{\infty} \cos(t\omega) f(t) \, dt. \]

3. The Fourier Sine Transform (FST):
   \[ F_s(\omega) = [\mathcal{F}_s f(t)](\omega) = \int_{-\infty}^{\infty} \sin(t\omega) f(t) \, dt. \]

4. The Laplace Transform (LT):
   \[ L(s) = [\mathcal{L} f(t)](s) = \int_{0}^{\infty} e^{-st} f(t) \, dt. \]
Note that
\[
[\mathcal{F} f (t)](\omega) = \int_{-\infty}^{\infty} [\cos (t \omega) + i \sin (t \omega)] f (t) \, dt = [\mathcal{F}_c f (t)](\omega) + i [\mathcal{F}_s f (t)](\omega).
\]

Thus the Fourier transform of a \textbf{real even function} is real and the Fourier transform of a \textbf{real odd function} is imaginary.

If \( s = i \omega \) and \( f (t) = 0 \) for \( t < 0 \), then
\[
[\mathcal{L} f (t)](i \omega) = \int_{0}^{\infty} e^{i \omega t} f (t) \, dt.
\]

Therefore the Laplace transform can be regarded as a special case of the Fourier transform.

The Fourier transform is a complex-valued function that decomposes a function of time (a signal) into the frequencies that make it up.

Fourier series are powerful tools in treating various problems involving periodic functions. Many practical problems do not involve periodic functions. Therefore, it is desirable to generalize the method of Fourier series to include non-periodic functions. If \( f \) is not periodic then we may regard it as periodic with an infinite period, i.e., we would like to see what happens if we let \( L \to \infty \). We shall do this for reasons of motivation as well as for making it plausible that for a non-periodic function, one should expect an integral representation (Fourier integral) instead of Fourier series.

Recall that if \( f \) is a periodic function with period \( 2L \) on \( \mathbb{R} \), then \( f \) has a Fourier Series (FS) representation of the form
\[
f (t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos \left( \frac{n \pi}{L} t \right) + b_n \sin \left( \frac{n \pi}{L} t \right) \quad (2),
\]
where
\[
\begin{align*}
a_n &= \frac{1}{L} \int_{-L}^{L} f (t) \cos \left( \frac{n \pi}{L} t \right) \, dt \quad n = 0; 1; 2; \ldots \\
b_n &= \frac{1}{L} \int_{-L}^{L} f (t) \sin \left( \frac{n \pi}{L} t \right) \, dt \quad n = 1; 2; 3; \ldots
\end{align*}
\]
Set \( \omega_n = \frac{n \pi}{L} \). Then (1) can be rewritten as:
\[
f (t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos (\omega_n t) + b_n \sin (\omega_n t) \quad (1),
\]
where
\[
\begin{align*}
a_n &= \frac{1}{L} \int_{-L}^{L} f (t) \cos (\omega_n t) \, dt \quad n = 0; 1; 2; \ldots \\
b_n &= \frac{1}{L} \int_{-L}^{L} f (t) \sin (\omega_n t) \, dt \quad n = 1; 2; 3; \ldots
\end{align*}
\]
Note that
\[
\Delta \omega = \omega_{n+1} - \omega_n = \frac{(n + 1) \pi}{L} - \frac{n \pi}{L} = \frac{\pi}{L}.
\]
Using $\Delta \omega$, we write the Fourier series in the form

$$f(t) = \frac{1}{L} \int_{-L}^{L} f(t) \, dt + \frac{1}{\pi} \sum_{n=0}^{\infty} \cos(\omega_n t) \Delta \omega \int_{-L}^{L} f(t) \cos(\omega_n t) \, dt$$

$$+ \frac{1}{\pi} \sum_{n=0}^{\infty} \sin(\omega_n t) \Delta \omega \int_{-L}^{L} f(t) \sin(\omega_n t) \, dt.$$ 

This representation is valid for any fixed $L$, arbitrary large, but finite. Letting $L \to \infty$ and assuming that the resulting non-periodic function is absolutely integrable over $(\infty, \infty)$, i.e., then

$$\int_{-\infty}^{\infty} |\tilde{f}(t)| < \infty,$$

where $\tilde{f}(t) = \lim_{L \to \infty} f(t)$. Thus the infinite series (1) becomes an integral from 0 to $\infty$, i.e.,

$$[\hat{\mathcal{F}} f(t)](\omega) = \frac{1}{\pi} \int_{0}^{\infty} [A(\omega) \cos(\omega t) + B(\omega) \sin(\omega t)] \, dt,$$

which is called the Fourier integral of $f$, where

$$A(\omega) = \int_{-\infty}^{\infty} \cos(\omega t) \tilde{f}(t) \, dt \quad \text{and} \quad B(\omega) = \int_{-\infty}^{\infty} \sin(\omega t) \tilde{f}(t) \, dt.$$

Fourier sine and cosine transforms are forms of the Fourier integral transform that do not use complex numbers. They are the forms originally used by Joseph Fourier and are still preferred in some applications, such as signal processing or statistics.

The Fourier cosine transform of a real function is the real part of the full complex Fourier transform. The cosine transform of an even function $f$ is necessarily an even function of frequency, i.e. for all $\omega$,

$$[\hat{\mathcal{F}}_c f(t)](-\omega) = [\hat{\mathcal{F}}_c f(t)](\omega).$$

Similarly, the sine transform of an odd function is an even function of frequency, i.e. for all $\omega$,

$$[\hat{\mathcal{F}}_s f(t)](-\omega) = -[\hat{\mathcal{F}}_s f(t)](\omega).$$

Clearly the cosine transform of an odd function is zero.

Some authors only define the cosine transform for even functions of $t$, in which case its sine transform is zero. Since cosine is also even, a simpler formula can be used,

$$[\hat{\mathcal{F}}_c f(t)](\omega) = 2 \int_{0}^{\infty} \cos(t \omega) f(t) \, dt.$$

For an odd function $f$, the sine transform may be defined as

$$[\hat{\mathcal{F}}_s f(t)](\omega) = 2 \int_{0}^{\infty} \sin(t \omega) f(t) \, dt.$$
Fourier Transform Pair. The Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{C}$ is
\[
[\mathcal{F} f (t)](\omega) = \int_{-\infty}^{\infty} e^{-i \omega t} f (t) \, dt.
\]

When the independent variable $t$ represents time, the transform variable $\omega$ represents frequency (e.g. if time is measured in seconds, then the frequency is in hertz). Under suitable conditions, $f$ is determined by $[\mathcal{F} f (t)]$ via the inverse transform:
\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \omega t} [\mathcal{F} f (t)] (\omega) \, d\omega,
\]
for any real number $t$.

The statement that $f$ can be reconstructed from $\mathcal{F}$ is known as the Fourier inversion theorem, and was first introduced in Fourier’s “Analytical Theory of Heat.” The functions $f$ and $[\mathcal{F} f (t)]$ often are referred to as a Fourier integral pair or Fourier transform pair.

Suppose that $f$ is an integrable function on $(0, 1)$. Define Fourier cosine transform $F_c$ and Fourier sine transform $F_s$ by
\[
F_c(\omega) = \int_{0}^{\infty} \cos (\omega t) f (t) \, dt; \quad F_s(\omega) = \int_{0}^{\infty} \sin (\omega t) f (t) \, dt.
\]

The inversion formulas for the Fourier cosine and Fourier sine transforms are
\[
f(t) = \frac{1}{2\pi} \int_{0}^{\infty} \cos (\omega t) F_c(\omega) \, d\omega; \quad F_s(t) = \frac{1}{2\pi} \int_{0}^{\infty} \sin (\omega t) F_s(\omega) \, d\omega.
\]

Properties of the Fourier transform. We assume $f$, $g$, and $h$ are Riemann integrable. Then the Fourier transform has the following properties:

1. Linearity. For any complex numbers $a$ and $b$, if $h(t) = af(t) + bg(t)$, then from linearity of the integral, we obtain
\[
[\mathcal{F} h (t)] (\omega) = a [\mathcal{F} f (t)] (\omega) + b [\mathcal{F} g (t)] (\omega).
\]

2. Time Shifting. For any real number $t_0$, if $h(t) = f(t - t_0)$, then
\[
[\mathcal{F} h (t)] (\omega) = e^{-i \omega t_0} [\mathcal{F} f (t)] (\omega).
\]

Proof. Let $\tau = t - t_0$, then
\[
\mathcal{F} h (\tau)] (\omega) = \int_{-\infty}^{\infty} e^{-i \omega (\tau + t_0)} f (\tau) \, d\tau
  = e^{-i \omega t_0} \int_{-\infty}^{\infty} e^{-i \omega \tau} f (\tau) \, d\tau
  = e^{-i \omega t_0} [\mathcal{F} f (t)] (\omega).
\]

3. Frequency Shifting. For any real number $\omega_0$, if $h(t) = e^{i \omega_0 t} f (t)$, then
\[
[\mathcal{F} h (t)] (\omega) = [\mathcal{F} f (t)] (\omega - \omega_0).
\]
Proof. By definition, we have

\[
[\mathcal{F} h(t)](\omega) = e^{i \omega_0 t} \int_{-\infty}^{\infty} e^{-i \omega t} f(t) \, dt \\
= \int_{-\infty}^{\infty} e^{-i (\omega - \omega_0) t} f(t) \, dt = [\mathcal{F} f(t)](\omega - \omega_0).
\]

\[\square\]

4. Time scaling. For a non-zero real number \( a \), if \( h(t) = f(at) \), then

\[ [\mathcal{F} h(t)](\omega) = \frac{1}{|a|} [\mathcal{F} f(t)](\omega/a). \]

The case \( a = -1 \) leads to the time-reversal property, which states: if \( h(t) = f(t) \), then

\[ [\mathcal{F} h](\omega) = [\mathcal{F} f(t)](-\omega). \]

Real and imaginary part in time.

- If \( h(t) = \Re(f(t)) \), then
  \[ [\mathcal{F} h(t)](\omega) = \frac{1}{2} \left[ [\mathcal{F} h f](\omega) + [\mathcal{F} f(t)](-\omega) \right]. \]

- If \( h(x) = \Im(f(t)) \), then
  \[ [\mathcal{F} h(t)](\omega) = \frac{1}{2i} \left[ [\mathcal{F} f(t)](\omega) - [\mathcal{F} f(t)](-\omega) \right]. \]

Integration. Substituting \( \omega = 0 \) in the definition, we obtain

\[ [\mathcal{F} f(t)](0) = \int_{-\infty}^{\infty} f(t) \, dt. \]

This means that the evaluation of the Fourier transform at the origin \( \omega = 0 \) equals the integral of \( f \) over all its domain.

Alternate Forms of Fourier Transforms Pairs. There are alternate forms of the Fourier
Transform that you may see in different references.

<table>
<thead>
<tr>
<th>Fourier transform:</th>
<th>$F(\omega) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) , dt$;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse Fourier transform:</td>
<td>$F^{-1}(\mathcal{F} f(t)(\omega)) = f(t) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) , dt$;</td>
</tr>
<tr>
<td>Fourier transform:</td>
<td>$F(\omega) = \int_{-\infty}^{\infty} e^{-2i\omega t} f(t) , dt$;</td>
</tr>
<tr>
<td>Inverse Fourier transform:</td>
<td>$F^{-1}(\mathcal{F} f(t)(\omega)) = f(t) = \int_{-\infty}^{\infty} e^{2\pi i\omega t} f(t) , dt$;</td>
</tr>
<tr>
<td>Fourier cosine transform:</td>
<td>$F_{c}(\omega) = \frac{2}{\pi} \int_{0}^{\infty} \cos(\omega t) f(t) , dt$;</td>
</tr>
<tr>
<td>Inverse Fourier cosine transform:</td>
<td>$F_{s}^{-1}(\mathcal{F}<em>{c} f(t)(\omega)) = f(t) = \int</em>{0}^{\infty} \cos(t\omega) F_{c}(\omega) , d\omega$;</td>
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<tr>
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