**Inverse of Triangular Matrices**

Given a lower triangular matrix $L = (l_{i,j})$, we denote by $D$, the diagonal part and $\widehat{L}$, strictly lower triangular parts of $L$. Hence

$$L = D + \widehat{L} = \begin{pmatrix} l_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & l_{2,2} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & l_{n,n} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ l_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & 0 \end{pmatrix}.$$

If $L$ is not singular, then

$$D^{-1} = \begin{pmatrix} 1/l_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & 1/l_{2,2} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1/l_{n,n} \end{pmatrix}$$

and the matrix

$$\mathcal{L} = D^{-1}L = D^{-1}[D + \widehat{L}] = I_n + D^{-1}\widehat{L} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ l_{2,1}/l_{2,2} & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ l_{n,1}/l_{n,n} & l_{n,2}/l_{n,n} & \cdots & l_{n,n-1}/l_{n,n} & 1 \end{pmatrix}.$$

Note that $\mathcal{L}^{-1} = L^{-1}\left[D^{-1}\right]^{-1} = L^{-1}D$; hence $L^{-1} = \mathcal{L}^{-1}D^{-1}$.

**Cayley-Hamilton's Theorem.** *If $K_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ is the characteristic polynomial of the $n \times n$ matrix $A$, then*

$$K_A(A) = A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI_n = Z_n,$$

*where $I_n$ is the $n \times n$ identity matrix and $Z_n$ the $n \times n$ zero matrix.*

If $A$ is invertible, then according to the Cayley Hamilton's theorem,

$$A(A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I_n) = -a_nI_n;$$

from this, we obtain

$$A^{-1} = \frac{1}{a_n}\left(A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I_n\right)$$

Since all the eigenvalues of the matrix $\mathcal{L}$ are equal to 1, the characteristic polynomial of $\mathcal{L}$ is as follows:

$$K_{\mathcal{L}}(\lambda) = (\lambda-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \lambda^{n-k}$$
$$= \lambda^n - \binom{n}{1}\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} \cdots\cdots (-1)^k \binom{n}{k}\lambda^{n-k} \cdots\cdots (-1)^{n-1}\binom{n}{n-1}\lambda + (-1)^n.$$

Now according to the Cayley-Hamilton's theorem, $K_A(A) = Z_n$, hence:

$$\mathcal{L}^{-1} = (-1)^{n-1}\left[\mathcal{L}^{n-1} - \binom{n}{1}\mathcal{L}^{n-2} \cdots\cdots (-1)^k \binom{n}{k}\mathcal{L}^{n-k-1} \cdots\cdots (-1)^{n-2}\binom{n}{2}\mathcal{L} + \binom{n}{1}(-1)^{n-1} I_n\right].$$

Similarly, we can find the inverse of an upper triangular matrix $U = D + \widehat{U}$, by finding first $\mathcal{U} = D^{-1}U = I_n + D^{-1}\widehat{U}$ and then using the following formula:

$$\mathcal{U}^{-1} = (-1)^{n-1}\left[\mathcal{U}^{n-1} - \binom{n}{1}\mathcal{U}^{n-2} \cdots\cdots (-1)^k \binom{n}{k}\mathcal{U}^{n-k-1} \cdots\cdots (-1)^{n-2}\binom{n}{2}\mathcal{U} + \binom{n}{1}(-1)^{n-1} I_n\right].$$

**Example.** Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 5 & 6 & 7 & 1 & 0 & 0 \\ 8 & 9 & -1 & -2 & 1 & 0 \\ -3 & -4 & -5 & -6 & -7 & 1 \end{pmatrix}.$$

To find $L^{-1}$, we need to find:

$$L^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 14 & 8 & 1 & 0 & 0 & 0 \\ 43 & 40 & 14 & 1 & 0 & 0 \\ 21 & 2 & -16 & -4 & 1 & 0 \\ -115 & -127 & -45 & 2 & -14 & 1 \end{pmatrix}, \qquad L^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 \\ 33 & 12 & 1 & 0 & 0 & 0 \\ 170 & 102 & 21 & 1 & 0 & 0 \\ -35 & -77 & -45 & -6 & 1 & 0 \\ -609 & -425 & -22 & 24 & -21 & 1 \end{pmatrix}$$

$$L^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 1 & 0 & 0 & 0 & 0 \\ 60 & 16 & 1 & 0 & 0 & 0 \\ 442 & 192 & 28 & 1 & 0 & 0 \\ -346 & -284 & -88 & -8 & 1 & 0 \\ -1576 & -562 & 162 & 60 & -28 & 1 \end{pmatrix}, \text{ and } L^5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 10 & 1 & 0 & 0 & 0 & 0 \\ 95 & 20 & 1 & 0 & 0 & 0 \\ 915 & 310 & 35 & 1 & 0 & 0 \\ -1210 & -675 & -145 & -10 & 1 & 0 \\ -2141 & 190 & 605 & 110 & -35 & 1 \end{pmatrix}.$$

The characteristic polynomial of $L$ is

$$K_L(\lambda) = (\lambda-1)^6 = \lambda^6 - 6\lambda^5 + 15\lambda^4 - 20\lambda^3 + 15\lambda^2 - 6\lambda + 1;$$

and since

$$K_L(L) = L^6 - 6L^5 + 15L^4 - 20L^3 + 15L^2 - 6L + I_6 = Z_6,$$

we conclude that

$$L^{-1} = -\left[L^5 - 6L^4 + 15L^3 - 20L^2 + 15L - 6I_6\right] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 \\ -28 & 22 & -7 & 1 & 0 & 0 \\ -41 & 31 & -13 & 2 & 1 & 0 \\ -435 & 333 & -128 & 20 & 7 & 1 \end{pmatrix}.$$