Equilibrium Points

♣ **Limit-Cycle**. A limit-cycle on a plane or a two-dimensional manifold is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches minus-infinity. Such behavior is exhibited in some nonlinear systems. In the case where all the neighboring trajectories approach the limit-cycle as time $t \to +\infty$, it is called a *stable* or *attractive* limit-cycle. If instead all neighboring trajectories approach it as time $t \to -\infty$, it is an *unstable* or *non-attractive* limit-cycle. In all other cases it is neither "stable" nor "unstable".

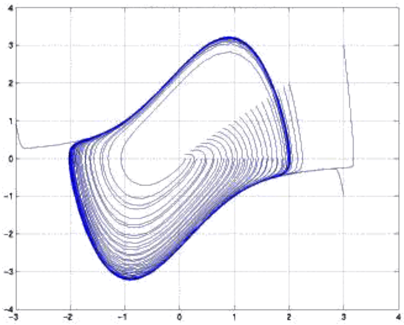


Figure illustrating a stable limit cycle for the Van der Pol oscillator

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0.$$

As seen in the figure, trajectories for various initial states of this system converge to the limit cycle.

♣ **Qualitative Analysis of Systems with Complex Eigenvalues**. Consider the linear homogeneous system

$$\begin{cases} x'(t) = ax + by, \\ y'(t) = cx + dy. \end{cases},$$

where the eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. Then the general solution is given by

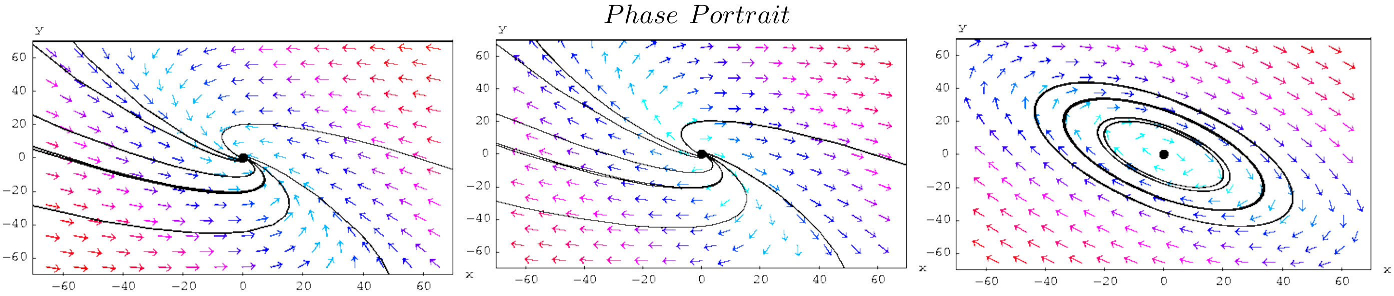
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} (\cos \beta t + i \sin \beta t) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

where $X_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is a complex eigenvector of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The fact that $\sin$ and $\cos$ are periodic function implies that the solutions spiral around the origin with a period of $2\pi/\beta$. This quantity is called the *natural period* of the system. Every solution takes this amount of time to wind once around the origin. The *natural frequency* is the reciprocal of the natural period, or $\beta/2\pi$. Moreover

- If $\alpha < 0$, then the solutions tend to the origin (when $t \to \infty$) while spiraling. In this case, the equilibrium point is called a *spiral sink*.
- If $\alpha > 0$, then the solutions explode or get away from the origin (when $t \to \infty$) while spiraling. In this case, the equilibrium point is called a *spiral source*.
- If $\alpha = 0$, then the solutions are periodic. This means that the trajectories are closed curves or cycles. In this case, the equilibrium point is called a *center*.

Phase Portrait



$\alpha < 0$ $\alpha > 0$ $\alpha = 0$

The question of "which way" the solutions spiral (clockwise or counterclockwise) can be answered by looking at the direction field. Even one vector of the vector field is enough to tell which way a system with complex eigenvalues spirals. For example, if the direction field at $(1, 0)$ points down into the fourth quadrant, then the solutions must spiral in the clockwise direction. If the direction field at $(1, 0)$ points up into the first quadrant, then the solutions must spiral in the counterclockwise direction.

♣ **Qualitative Analysis of Systems with Zero as an Eigenvalue**. Consider a linear system where at least one of the eigenvalues is zero. In fact, it is easy to see that this happen if and only if we have more than one equilibrium point (which is $(0, 0)$). In this case, we will have a line of equilibrium points (the direction vector for this line is the eigenvector associated to the eigenvalue zero).

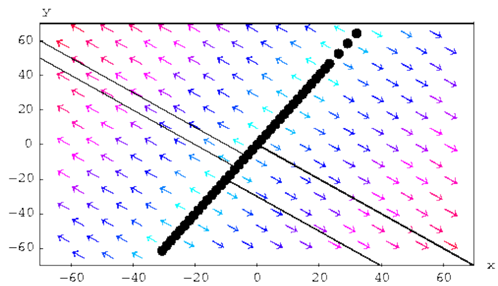
Consider the linear homogeneous system

$$\begin{cases} x'(t) = 2x - y, \\ y'(t) = -2x + y. \end{cases}.$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$. The associated eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively. Therefore the general solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{0t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Note that all the solutions are line parallel to the line $x + y = 0$. When $t \to \infty$, the trajectory goes to infinity. But when $t \to -\infty$, the trajectory converge to the equilibrium point on the line of equilibrium points $2x - y = 0$. The picture below explains more what is happening.



♣ **When Linearization Fails**. Unfortunately, in some cases the information given by the linearized system is not enough to completely determine the behavior of the nonlinear system near the equilibrium point. We shall explain this problem with the help of two examples.

Consider the systems:

$$(a)\quad \begin{cases} x'(t) = y - (x^2+y^2)x, \\ y'(t) = -x - (x^2+y^2)y. \end{cases} \qquad (b)\quad \begin{cases} x'(t) = y + (x^2+y^2)x, \\ y'(t) = -x + (x^2+y^2)y. \end{cases}$$

Both systems have the same unique equilibrium point $(0,0)$ and the same linearized system:

$$(c)\quad \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The eigenvalues of the linearized system (c) are $\lambda_1 = i$ and $\lambda_2 = -i$. Since the real part of the eigenvalues is zero, the equilibrium point $(0,0)$ of (c) is a center. All nonzero solutions of the linearized system lie on periodic solutions that wind around $(0,0)$ in the clockwise direction. But there are no periodic solution for the nonlinear systems (a). To see this, we regard the vector field as a sum of two vector fields. The linear field $(y,-x)$ and the nonlinear vector field $(-(x^2+y^2)x, -(x^2+y^2)y)$. The linear field $(y,-x)$ is always tangent to circles centered at the origin. But the vector field $(-(x^2+y^2)x, -(x^2+y^2)y)$ always point toward $(0,0)$, since this vector field is basically a multiple of $(-x,-y)$ by a positive number x^2+y^2. The net effect of adding these two vector fields is a vector field that always point "into" a circle centered at the origin. This forces solutions to spiral slowly into the equilibrium point $(0,0)$ rather than to circulate on periodic orbit.

Note that the system (b) is obtained from (a) by changing the signs of its higher order terms, so solutions of (b) spiral away from the origin.

The solutions of the nonlinear systems (a) and (b), and the solutions of the linearized system (c) are still approximately the same, at least for a short amount of time. The problem is that because the fixed point of the linearized system is a center,any small perturbation caused by the inclusion of the nonlinear terms can turn the center into a spiral sink or a spiral source.

Fortunately there are only two situations in which the long term behavior of solutions near an equilibrium point of the nonlinear system and its linearization can differ. One is when the linearized system is a center, the other is when the linearized system has zero as eigenvalue. In any other case, the local picture of the nonlinear system near an equilibrium point looks like its linearization.

♣ **The Distance Formula**. Suppose $e = (0,0)$ is an equilibrium point of the following system of differential equations

$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)). \end{cases}$$

Let $p(t) = (x(t), y(t)) \neq (0,0) = e$ be a solution point. The distance between $p(t)$ and e is given by

$$d(p(t), e) = \sqrt{[x(t)]^2 + [y(t)]^2}$$

and its instantaneous rate of change is obtain as follows:

$$[d(p(t), e)]' = \frac{1}{2}\left[\frac{2x(t)x'(t) + 2y(t)y'(t)}{\sqrt{[x(t)]^2 + [y(t)]^2}}\right] = \frac{x(t)\,[f(x(t), y(t))] + y(t)\,[g(x(t), y(t))]}{\sqrt{[x(t)]^2 + [y(t)]^2}}$$

let $D(t) = x(t)\,[f(x(t), y(t))] + y(t)\,[g(x(t), y(t))]$, then since $\sqrt{[x(t)]^2 + [y(t)]^2} > 0$, the distance between $p(t)$ and e is increasing (resp. decreasing) if $D(t) > 0$ (resp. $D(t) < 0$).

If the equilibrium point (a,b) is different from $(0,0)$, then by choosing $u = x-a$ and $v = y-b$, we obtain a system, where $(0,0)$ is an equilibrium point.

Examples. Consider the following nonlinear differential equations systems:

$$(a)\quad \begin{cases} x' = & -x^3 - y \\ y' = x - y^3 \end{cases} \quad \text{and} \quad (b)\quad \begin{cases} x' = y - xy^2 \\ y' = & -x + x^2 y \end{cases}.$$

The origin $e = (0,0)$ is the equilibrium point of both (a) and (b). The linearized systems around the origin are:

$$(a)\quad \begin{cases} x' = & -y \\ y' = x \end{cases} \quad \text{and} \quad (b)\quad \begin{cases} x' = y \\ y' = & -x \end{cases}.$$

The characteristic equation of both systems is $\lambda^2 + 1 = 0$ and the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. Thus the equilibrium point e of the linearized systems is a center. Notice that

$$(a)\quad D(t) = x(-y) + yx = 0 \quad \text{and} \quad (b)\quad D(t) = xy + y(-x) = 0;$$

hence the origin is a center and the trajectories are circles. This is a case when the linearizations fails to describe the solutions of the original systems around the origin. Using the nonlinear systems, we find that

(a) $D(t) = (-x^4 - xy) + (xy - y^4) = -(x^4 + y^4) < 0$. This says that the distance between points on the trajectory and the origin is decreasing with time; that is, the trajectory spiral toward the origin.

(b) $D(t) = (xy - x^2y^2) + (-yx + x^2y^2) = 0$. Thus all trajectories are circles around the origin. But there are also other equilibrium points; in fact any point on the hyperbola $y = \dfrac{1}{x}$ is an equilibrium point; so for $a \neq 0$, $(a, 1/a)$ is an equilibrium point. By choosing $u = x - a$ and $v = y - 1/a$, the system becomes:

$$\begin{cases} u' = (v - \dfrac{1}{a}) \left[\dfrac{1}{a}u + av - uv\right] \\ v' = & -(u - a) \left[\dfrac{1}{a}u + av - uv\right] \end{cases}.$$

The linearized system is

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{bmatrix} -\frac{1}{a^2} & -1 \\ 1 & a^2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = a^2 - 1/a^2$. This is the case when the linearized system fails to describe the behavior of the phase plane of the original system. But the distance formula tells us that every trajectory is a circle centered around the origin with two stationary points at $(a, 1/a)$ on the circle.

♣ **Bifurcation Point**. When a nonlinear system depends on a parameter, then as the parameter changes, the equilibrium points can change. That is, as the parameter changes a bifurcation can occur. Consider the one-parameter system:

$$\begin{cases} x'(t) = x^2 - \alpha \\ y'(t) = -(x^2 + 1)y. \end{cases}, \quad \text{where } \alpha \text{ is a parameter.}$$

- If $\alpha < 0$, then the there is no x-nullclines, hence the system has no equilibrium points.
- If $\alpha = 0$, then the system has exactly one equilibrium point at $(0,0)$.
- If $\alpha > 0$, then the system has two equilibrium points $(-\sqrt{\alpha}, 0)$ and $(\sqrt{\alpha}, 0)$.

The system changes from having no fixed points to having two fixed points when the parameter α is increase through $\alpha = 0$. We say that the system has a bifurcation when $\alpha = 0$ and that α is a bifurcation point.

The Jacobian matrix is $J = \begin{bmatrix} 2x & 0 \\ -2xy & -x^2 - 1 \end{bmatrix}$ with

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad J(-\sqrt{\alpha}, 0) = \begin{bmatrix} -2\sqrt{\alpha} & 0 \\ 0 & -\alpha - 1 \end{bmatrix}, \quad \text{and} \quad J(\sqrt{\alpha}, 0) = \begin{bmatrix} 2\sqrt{\alpha} & 0 \\ 0 & -\alpha - 1 \end{bmatrix}.$$

- For $\alpha = 0$, there will be a line equilibrium (since one of the eigenvalues is zero).
- For $\alpha > 0$, the point $(-\sqrt{\alpha}, 0)$ is a sink and $(\sqrt{\alpha}, 0)$ is a saddle point.

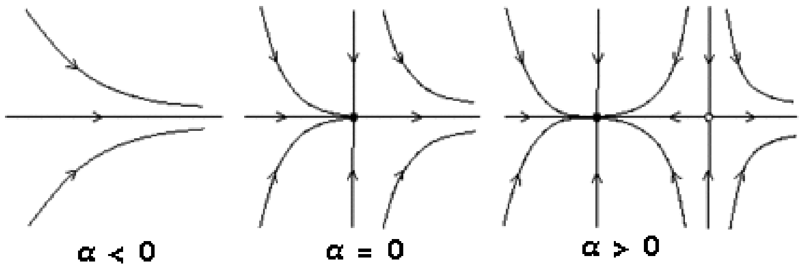
Saddle-Node Bifurcation

A *saddle-node bifurcation* or *tangent bifurcation* is a collision and disappearance of two equilibria in dynamical systems. In autonomous systems, this occurs when the critical equilibrium has one zero eigenvalue. This phenomenon is also called *fold* or *limit point bifurcation.*

In n-dimensional case with $n \geq 2$, the Jacobian matrix at the saddle-node bifurcation has

- a simple zero eigenvalue $\lambda_1 = 0$, as well as
- n_r eigenvalues with $\lambda_j < 0$, and n_s eigenvalues with $\lambda_j > 0$, with $n_r + n_s + 1 = n$.

Thus in the above example, for $\alpha = 0$, there is a saddle-node bifurcation point at $(0, 0)$.



Andronov-Hopf Bifurcation

A *Hopf* or *Poincare-Andronov-Hopf* bifurcation is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane.

Consider the one-parameter systems:

$$(a) \quad \begin{cases} x'(t) = \beta x - y - (x^2 + y^2)x, \\ y'(t) = x + \beta y - (x^2 + y^2)y. \end{cases} \qquad (b) \quad \begin{cases} x'(t) = \beta x - y + (x^2 + y^2)x, \\ y'(t) = x + \beta y + (x^2 + y^2)y. \end{cases}$$

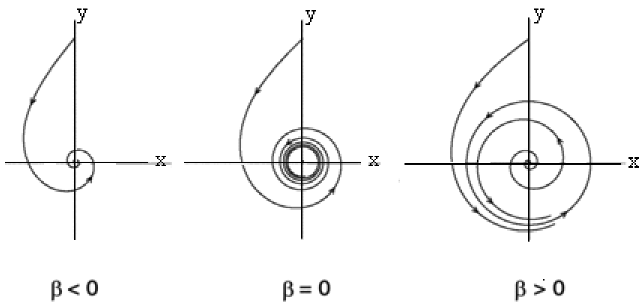
Both systems have the same unique equilibrium point at the origin and the same linearized system:

$$(c) \quad \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \beta & -1 \\ 1 & \beta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

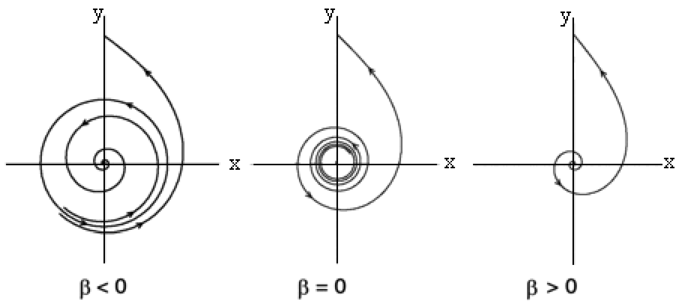
The eigenvalues of the linearized system (c) are $\lambda_1 = \beta + i$ and $\lambda_2 = \beta - i$.

- The equilibrium point of the system (a) is asymptotically stable for $\beta < 0$, weakly stable at $\beta = 0$, and unstable for $\beta > 0$. Moreover, there is a unique and stable circular limit cycle that exists for $\beta > 0$ and has radius $\sqrt{\beta}$. This is a *supercritical* Andronov-Hopf bifurcation.

- The origin is asymptotically stable for $\beta < 0$ and unstable for $\beta \geq 0$, and weakly stable $\beta = 0$. Moreover, there is a unique and unstable limit cycle for $\beta < 0$. This is a *subcritical* Andronov-Hopf bifurcation.



Supercritical Hopf bifurcation



Subcritical Hopf bifurcation