Nonlinear Systems of Ordinary Differential Equations

♣ **Dynamical System**. A dynamical system has a state determined by a collection of real numbers, or more generally by a set of points in an appropriate state space. Small changes in the state of the system correspond to small changes in the numbers.

The evolution rule of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic: for a given time interval only one future state follows from the current state.

The mathematical models used to describe the swinging of a clock pendulum, the flow of water in a pipe, or the number of fish each spring in a lake are examples of dynamical systems.

♣ **Autonomous System**. An autonomous differential equation is a system of ordinary differential equations which does not depend on the independent variable. It is of the form

$$\frac{d}{dt}X(t) = F(X(t)),$$

where X takes values in n-dimensional Euclidean space and t is usually time.

It is distinguished from systems of differential equations of the form

$$\frac{d}{dt}X(t) = G(X(t), t),$$

in which the law governing the rate of motion of a particle depends not only on the particle's location, but also on time; such systems are not autonomous.

Autonomous systems are closely related to dynamical systems. Any autonomous system can be transformed into a dynamical system and, using very weak assumptions, a dynamical system can be transformed into an autonomous systems.

♣ **Jacobian Matrix**. Consider the function $F : I\!R^n \longrightarrow I\!R^m$, where

$$F(x_1, x_2, \ldots, x_n) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \;\vdots\;\; \vdots \;\; \vdots \;\; \vdots\; \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}.$$

The partial derivatives of $f_1(x_1, ..., x_n), \ldots, f_m(x_1, ..., x_n)$ (if they exist) can be organized in an $m \times n$ matrix.

The Jacobian matrix of $F(x_1, x_2, \ldots, x_n)$ denoted by J_F is as follows:

$$J_F(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

Its importance lies in the fact that it represents the best linear approximation to a differentiable function near a given point.

♣ **Qualitative Analysis**. Very often it is almost impossible to find explicitly or implicitly the solutions of a system (specially nonlinear ones). The qualitative approach as well as numerical one are important since they allow us to make conclusions regardless whether we know or not the solutions.

Nullclines and Equilibrium Points

Consider the system of first order ordinary differential equations:

$$\begin{cases} x_1' = f_1(x_1, x_2 \ldots, x_n) \\ x_2' = f_2(x_1, x_2 \ldots, x_n) \\ \quad \vdots \qquad \vdots \qquad \vdots \\ x_n' = f_n(x_1, x_2 \ldots, x_n). \end{cases}$$

The x_j-*nullcline* is the set of points which satisfy $f_j(x_1, x_2, \ldots, x_n) = 0$. The intersection point of all the nullclines is called an *equilibrium point or fixed point* of the system.

The Jacobian matrix with constant entries, is identified with the matrix of a linear systems. Near a fixed point $(x_1^*, x_2^*, \ldots, x_n^*)$, the dynamics of the nonlinear system are qualitatively similar to the dynamics of the linear system associated with the Jacobian matrix $J(x_1^*, x_2^*, \ldots, x_n^*)$, provided its eigenvalues $\lambda_j' s$ have nonzero real parts. Fixed points with a Jacobian matrix such that $Re(\lambda_j) \neq 0$ are called *hyperbolic* fixed points. Otherwise, they are *non-hyperbolic* fixed points, whose stabilities must be determined directly.

Example 1. Consider the system:

$$\begin{cases} x'(t) = x(1-x) - xy, \\ y'(t) = 2y(1 - \frac{y^2}{2}) - 3x^2 y. \end{cases}$$

The x-nullclines are given by
$x'(t) = x(1-x) - xy = 0$
which is equivalent to
$x = 0$ or $y = -x + 1$.
The y-nullclines are given by
$y'(t) = 2y(1 - \frac{y}{2}) - 3xy = 0$
which is equivalent to
$y = 0$ or $3x^2 + y^2 = 2$.

Example 2. Consider the model describing two competing species:

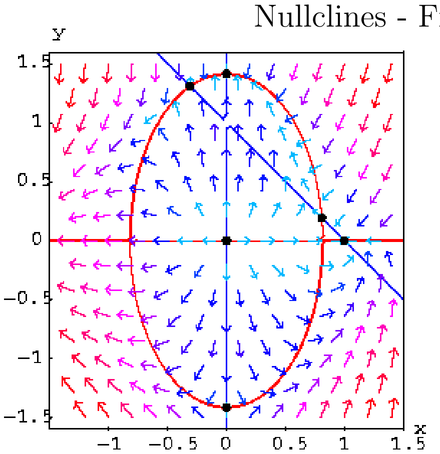
$$\begin{cases} x'(t) = x(1 - x - y), \\ y'(t) = 2y\left(1 - \frac{y}{2} - \frac{3}{2}x\right). \end{cases}$$

The x-nullclines are
$x = 0$ or $y = -x + 1$.
The y-nullclines are
$y = 0$ or $y = -3x + 2$.

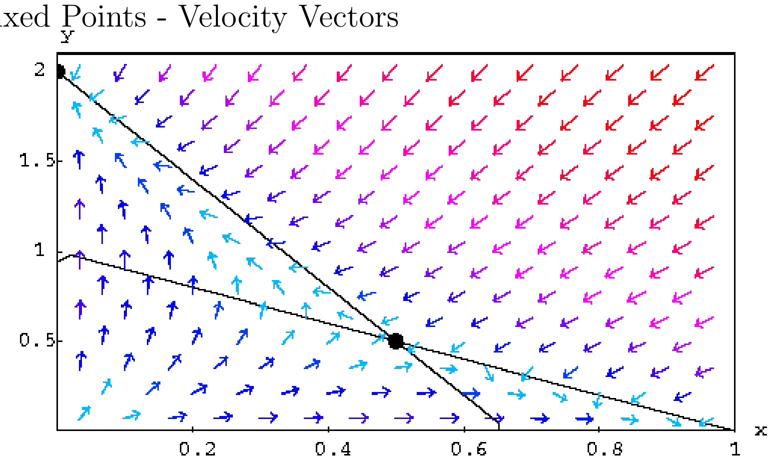
The equilibrium points are
$(0,0)$, $(0,2)$, $(1,0)$, and $(\frac{1}{2}, \frac{1}{2})$.

The components of the velocity vectors are $x'(t)$ and $y'(t)$. These vectors give the direction of the motion along the trajectories. We have the four natural directions (left, right, up, and down) and the other four directions (left-down, left-up, right-down, and right-up). These directions are obtained by looking at the signs of $x'(t)$ and $y'(t)$ and whether they are equal to 0. If both are zero, then we have an equilibrium point. Note that along the x-nullcline the velocity vectors are vertical while along the y-nullcline the velocity vectors are horizontal. Note that as long as we are traveling along a nullcline without crossing an equilibrium point, then the direction of the velocity vector must be the same. Once we cross an equilibrium point, then we may have a change in the direction (from up to down, or right to left, and vice-versa).

Nullclines - Fixed Points - Velocity Vectors



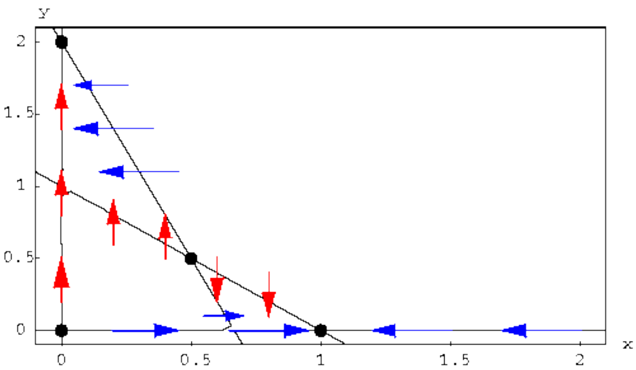
Example 1.



Example 2.

In order to find the direction of the velocity vectors along the nullclines, we pick a point on the nullcline and find the direction of the velocity vector at that point. The velocity vector along the segment of the nullcline delimited by equilibrium points which contains the given point will have the same direction. For example, consider the point $(1/3, 1)$ on the y-nullcline $y = -3x + 2$ in the second example. The velocity vector at this point is $(-1/9, 0)$. Therefore the velocity vector at any point on the line $y = -3x + 2$, with $x > 1/3$, is horizontal and points to the left (since $x' = -1/9 < 0$).

The picture below gives the nullclines and the velocity vectors along them.



Velocity Vectors

Remark. The point $(0, 0)$ is a fixed point of any linear system of ordinary differential equation, but a nonlinear system may have neither fixed points nor nullclines.

Example 3.

$$\begin{cases} x'(t) = x^2 + 1, \\ y'(t) = x(y - 1). \end{cases}$$

No x-nullcline.
The y-nullcline are $x = 0$ or $y - 1$.

No fixed point.

Example 4.

$$\begin{cases} x'(t) = x^2 + y^2 - 1, \\ y'(t) = x - y + 2. \end{cases}$$

The x-nullcline is the unit circle.
The y-nullcline is the line $y = x + 2$.

The nullclines do not intersect.

Nonlinear Autonomous Systems of Two Equations

Most of the interesting differential equations are non-linear and, with a few exceptions, cannot be solved exactly. Approximate solutions are arrived at using computer approximations.
A first order nonlinear autonomous system is:

$$\begin{cases} x'(t) = F(x,y), \\ y'(t) = G(x,y). \end{cases}$$

At the site:

http://cs.jsu.edu/mcis/faculty/leathrum/Mathlets/diffeq2.html

they use Java to show you graphs of solutions of first order nonlinear autonomous systems of two equations. To see the graphs of the vector field and flow curves go to

http://cs.jsu.edu/mcis/faculty/leathrum/Mathlets/vecfield.html

Here are a few examples of second order nonlinear autonomous systems:

Equation of motion of point mass in the (x,y)-plane under gravitational force:

$$x''_{tt} = kxr^3, \;\; y''_{tt} = kyr^3, \;\text{ where } r = \sqrt{x^2+y^2}.$$

Equation of motion of a point mass in the (x,y)-plane under central force:

$$x''_{tt} = xf(r), \;\; y''_{tt} = yf(r), \;\text{ where } r = \sqrt{x^2+y^2}.$$

Equations of motion of a projectile:

$$x''_{tt} = -f(y)g(v)x'_t, \;\; y''_{tt} = -f(y)g(v)y'_t - a, \;\text{ where } v = \sqrt{(x')^2+(y')^2}.$$

Linearization Technique

Consider the autonomous nonlinear system

$$\begin{cases} x'(t) = F(x,y), \\ y'(t) = G(x,y). \end{cases}$$

with (x^*, y^*) a fixed point. We would like to find the closest linear system when (x, y) is close to (x^*, y^*). In order to do that we need to approximate the functions $F(x, y)$ and $G(x, y)$ around the equilibrium point (x^*, y^*) by its tangent around that fixed point. From multi-variable calculus, we know that when (x, y) is close to (x^*, y^*), the nonlinear system may be approximated by the system

$$\begin{cases} \dfrac{d}{dt}\, x(t) = F(x,y) \approx F(x^*,y^*) + \dfrac{\partial F}{\partial x}(x^*,y^*)(x-x^*) + \dfrac{\partial F}{\partial y}(x^*,y^*)(y-y^*) \\ \dfrac{d}{dt} y(t) = G(x,y) \approx G(x^*,y^*) + \dfrac{\partial G}{\partial x}(x^*,y^*)(x-x^*) + \dfrac{\partial G}{\partial y}(x^*,y^*)(y-y^*). \end{cases}$$

Since (x^*, y^*) is a fixed point, we have $F(x^*, y^*) = G(x^*, y^*) = 0$. Thus

$$\begin{cases} \dfrac{d}{dt}\, x(t) \approx \dfrac{\partial F}{\partial x}(x^*,y^*)(x-x^*) + \dfrac{\partial F}{\partial y}(x^*,y^*)(y-y^*) \\ \dfrac{d}{dt} y(t) \approx \dfrac{\partial G}{\partial x}(x^*,y^*)(x-x^*) + \dfrac{\partial G}{\partial y}(x^*,y^*)(y-y^*). \end{cases}$$

This is a linear system. Its coefficient matrix is

$$J = \begin{bmatrix} \frac{\partial F}{\partial x}(x^*, y^*) & \frac{\partial F}{\partial y}(x^*, y^*) \\ \frac{\partial G}{\partial x}(x^*, y^*) & \frac{\partial G}{\partial y}(x^*, y^*) \end{bmatrix}.$$

This matrix is just the Jacobian matrix of the system at the fixed point (x^*, y^*). Thus

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x^*, y^*) & \frac{\partial F}{\partial y}(x^*, y^*) \\ \frac{\partial G}{\partial x}(x^*, y^*) & \frac{\partial G}{\partial y}(x^*, y^*) \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}.$$

Note. If the equilibrium point $(x^*, y^*) \neq (0,0)$, then by choosing $u = x - x^*$ and $v = y - y^*$, we may the system to a new system with $(0,0)$ as a fixed point.

Topological Classification

Linear stability analysis works for a hyperbolic fixed points. The nonlinear system's phase portrait near the fixed point is topologically unchanged due to small perturbations, and its dynamics are structurally stable or robust.

Poincare-Lyapunov Theorem. *If the eigenvalues of the Jacobian matrix evaluated at the fixed point are not equal zero or are not pure imaginary numbers, then the trajectories of the system around the equilibrium point behave the same way as the trajectories of the associated linear system.*

1. *If the eigenvalues are negative or complex with negative real part, then the fixed point is a sink (that is all the solutions will die at the equilibrium point). Note that if the eigenvalues are complex, then the solutions will spiral around the equilibrium point.*
2. *If the eigenvalues are positive or complex with positive real part, then the fixed point is a source (this means that the solutions on the trajectories will move away from the equilibrium point). Note that if the eigenvalues are complex, then the solutions will spiral away from the fixed point.*
3. *If the eigenvalues are real number with different sign (one positive and one negative), then the equilibrium point is a saddle point. In fact, there will be two solutions which approach the equilibrium point as $t \to \infty$, and two more solutions which approach the equilibrium point as $t \to -\infty$. For the linear system theses solutions are lines, but for the nonlinear system they are not in general. These four solutions are called separatrix.*

Let $p = trace[J(x^*, y^*)]$ and $q = \det[J(x^*, y^*)]$, then hyperbolic fixed points are classified as follows:

Repellers (Sources)	Unstable	$p > 0, \quad q > 0$	$Re(\lambda_1) > 0, \quad Re(\lambda_2) > 0$
Attractors (Sinks)	Stable	$p < 0, \quad q > 0$	$Re(\lambda_1) < 0, \quad Re(\lambda_2) < 0$
Saddle Points	Unstable	$q < 0$	$Re(\lambda_1) < 0, \quad Re(\lambda_2) > 0$

Linear stability analysis may fail for a non-hyperbolic fixed point:

$$Re(\lambda_1) = 0 \text{ and } Re(\lambda_2) = 0 \text{ or at least one eigenvalue is zero.}$$

The classifications for the fixed points of a nonlinear system are summarized in the following diagram:

p

Saddle points	Repellers (Sources)	q
Saddle points	Attractors (Sinks)	

Example 5. Consider the nonlinear system

$$\begin{cases} x'(t) = F(x,y) = x^3 - x, \\ y'(t) = G(x,y) = -2y. \end{cases}$$

The solution is:

$$\int \frac{dx}{x^3 - x} = \int dt \qquad \Rightarrow \quad x(t) = \pm\sqrt{(1 - C_1 e^{2t})^{-1}} \;;$$
$$\int \frac{dy}{y} = \int -2\,dt \qquad \Rightarrow \quad y(t) = C_2 e^{-2t}.$$

The fixed points are the intersections of the nullclines $y = 0$ (the x-*axis*) with

$$x = -1, \; x = 0, \; \text{and} \quad x = 1.$$

Stability at Fixed Points

The Jacobian matrix is $J(x,y) = \begin{bmatrix} 3x^2 - 1 & 0 \\ 0 & -2 \end{bmatrix}$ with

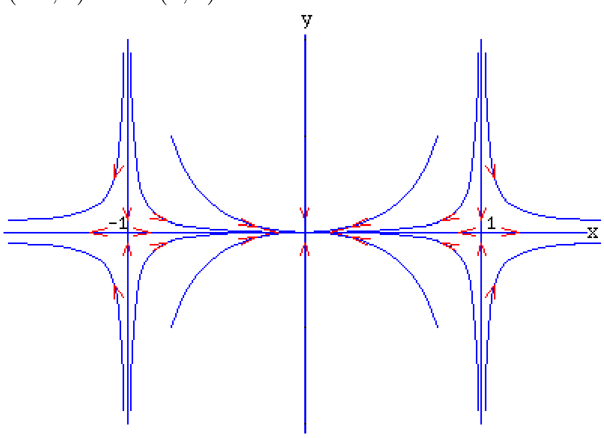
$$J(-1,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \text{and} \quad J(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Note that around the fixed points $(-1,0)$, $(0,0)$, and $(1,0)$, the nonlinear system should behave like the linear systems:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x+1 \\ y \end{bmatrix}, \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

respectively.

Since all the eigenvalues have nonzero real part, we conclude that all three fixed points are hyperbolic. Consequently, the nonlinear system has a stable node (attractor) at $(0,0)$ and saddle points at $(-1,0)$ and $(1,0)$.



Phase Portrait

Example 6. Consider the following second order nonlinear equation known as *Van der Pol* equation

$$\frac{d^2x}{dt^2} - (1-x^2)\frac{dx}{dt} + x = 0.$$

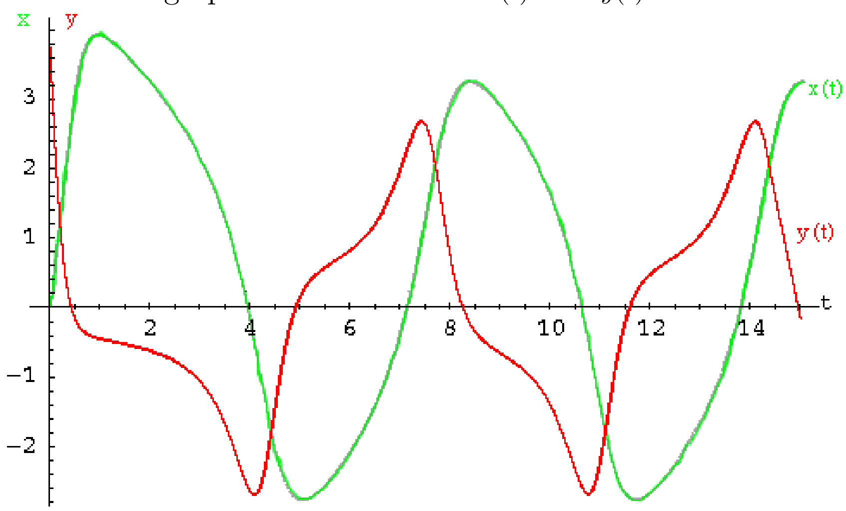
This can be translated into the following system. Set $y = \dfrac{dx}{dt}$. Then we have

$$\begin{cases} x'(t) = y, \\ y'(t) = -x + (1-x^2)y. \end{cases}$$

The x-*nullcline* is given by $\dfrac{dx}{dt} = y = 0$. Hence the x-*nullcline* is the x-*axis*.
The y-*nullcline* is given by $\dfrac{dy}{dt} = -x + (1-x^2)y$. Hence the y-*nullcline* is the curve $y = \dfrac{x}{1-x^2}$.
Thus $(x^*, y^*) = (0,0)$ is the only fixed point. The Jacobian matrix is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -1-2xy & 1-x^2 \end{bmatrix} \quad \text{with} \quad J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Next picture shows the graphs of the solutions $x(t)$ and $y(t)$ for the initial value $(0,4)$.



The linear system close to the original nonlinear system around the fixed point $(0,0)$ is

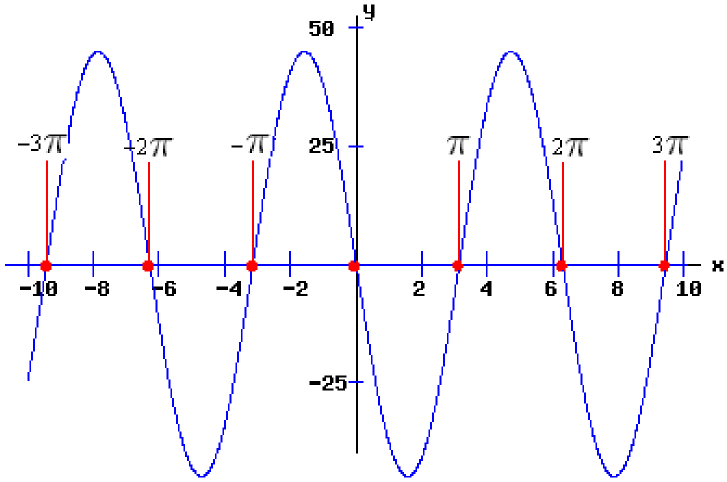
$$\begin{cases} x'(t) = y, \\ y'(t) = -x + y. \end{cases} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of this system are $\dfrac{1 \pm \sqrt{3}\, i}{2}$. Since the real part is positive, the solutions of the linear system spiral away from the origin.
Example 7. Finally, consider the following problem:

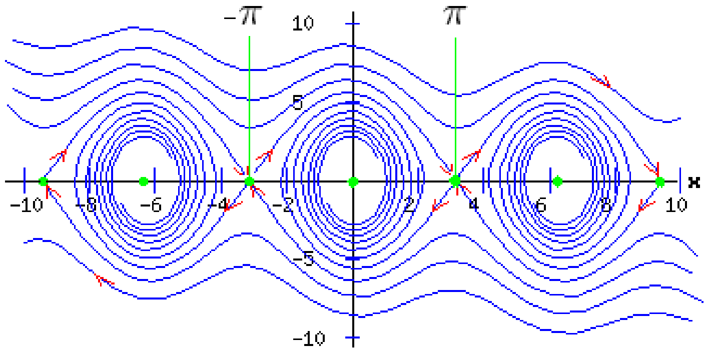
$$\begin{cases} x' = y \\ y' = -9\sin x - \dfrac{y}{5} \end{cases}$$

with $J = \begin{bmatrix} 0 & 1 \\ -9\cos x & -\frac{1}{5} \end{bmatrix}$ x-nullcline: $y = 0$ fixed points: $(n\pi, 0)$

y-nullclines: $y = -45\sin x$ $n = \cdots, -3, -2, 0, 1, 2, 3, \cdots$



Nullclines and Fixed Points



Phase Portrait