Partial Fractions

By memorizing the following formula:

$$\frac{1}{(s+a)(s+b)} = \frac{1}{(b-a)}\left[\frac{1}{s+a} - \frac{1}{s+b}\right],$$

we may avoid tedious calculations for changing certain fractions into partial fractions.

Here are some examples:

1. $$\frac{1}{(s+a)(s^2+b)} = \frac{s-a}{(s^2-a^2)(s^2+b)} = \frac{1}{b+a^2}\left[\frac{1}{s+a} - \frac{s-a}{s^2+b}\right]$$

2. $$\frac{1}{(s+a)(s^3+b)} = \frac{s^2-as+a^2}{(s^3+a^3)(s^3+b)} = \frac{1}{b-a^3}\left[\frac{1}{s+a} - \frac{s^2-as+a^2}{s^3+b}\right]$$

3. $$\frac{1}{(s^2+as+a^2)s} = \frac{(s-a)s^2}{(s^3-a^3)s^3} = \frac{1}{a^3}\left[\frac{s^2}{s^2+as+a^2} - \frac{s-a}{s}\right] = \frac{1}{a^2}\left[\frac{1}{s} - \frac{s-a}{s^2+as+a^2}\right]$$

4. $$\frac{1}{(s+a)(s+b)^2} = \frac{1}{(b-a)}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]\frac{1}{(s+b)}$$
$$= \frac{1}{(b-a)^2}\left[\frac{1}{s+a} - \frac{1}{s+b} - \frac{b-a}{(s+b)^2}\right]$$

5. $$\frac{1}{(s+a)(s+b)(s+c)} = \frac{1}{(b-a)}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]\frac{1}{(s+c)}$$
$$= \frac{1}{(b-a)(c-a)(c-b)}\left[\frac{c-b}{s+a} - \frac{c-a}{s+b} + \frac{b-a}{s+c}\right]$$

If the numerator of the fraction is not a constant, then in order to use the shortcut formula, we should try to get a fraction out of it which has a constant in the numerator. For example, the fraction

$$\frac{s+c}{(s+a)(s+b),}$$

should be change into

$$\frac{(s+a)+(c-a)}{(s+a)(s+b)} = \frac{1}{s+b} + \frac{c-a}{b-a}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] = \frac{1}{b-a}\left[\frac{c-a}{s+a} - \frac{c-b}{s+b}\right].$$

It is not advantageous to use the shortcut for the fraction:

$$\frac{1}{(s^2+a)(s^3+b)}$$

♣ **Examples**

(*a*)
$$\frac{6}{(x+2)(x+5)} = \frac{6}{(5-2)}\left[\frac{1}{x+2} - \frac{1}{x+5}\right] = \frac{2}{x+2} - \frac{2}{x+5} \ .$$

(*b*)
$$\begin{aligned}\frac{x+2}{(x+1)(x^2+4)} &= \frac{(x+1)+1}{(x+1)(x^2+4)} = \frac{1}{x^2+4} + \frac{1}{(x+1)(x^2+4)} \\ &= \frac{1}{x^2+4} + \frac{x-1}{(x^2-1)(x^2+4)} \\ &= \frac{1}{x^2+4} + \frac{x-1}{4-(-1)}\left[\frac{1}{x^2-1} - \frac{1}{x^2+4}\right] \\ &= \frac{1}{x^2+4} + \frac{1}{5}\left[\frac{1}{x+1} - \frac{x-1}{x^2+4}\right] \\ &= \frac{1}{5}\left[\frac{1}{x+1} - \frac{x-6}{x^2+4}\right].\end{aligned}$$

(*c*)
$$\begin{aligned}\frac{1}{(x-1)(x^3+5)} &= \frac{x^2+x+1}{(x^3-1)(x^3+5)} = \frac{x^2+x+1}{5-(-1)}\left[\frac{1}{x^3-1} - \frac{1}{x^3+5}\right] \\ &= \frac{1}{6}\left[\frac{1}{x-1} - \frac{x^2+x+1}{x^3+5}\right].\end{aligned}$$

(*d*)
$$\frac{16}{(x-1)(x+3)^2} = 4\left[\frac{1}{x-1} - \frac{1}{x+3}\right]\frac{1}{(x+3)} = \left[\frac{1}{x-1} - \frac{1}{x+3} - \frac{4}{(x+3)^2}\right].$$

(*e*)
$$\frac{5}{(x+1)(x+2)(x+4)} = \frac{5}{x+1}\left[\frac{1}{2}\left[\frac{1}{x+2} - \frac{1}{x+4}\right]\ \right] = \frac{5}{2}\left[\frac{1}{(x+1)(x+2)} - \frac{1}{(x+1)(x+4)}\right]$$

$$\begin{aligned} &= \frac{5}{2}\left[\frac{1}{x+1} - \frac{1}{x+2}\right] - \frac{5}{6}\left[\frac{1}{x+1} - \frac{1}{x+4}\right] \\ &= \frac{5}{6}\left[\frac{2}{x+1} - \frac{3}{x+2} + \frac{1}{x+4}\right].\end{aligned}$$