Pivoting Strategies For Solving Linear Systems

♣ **Gaussian Elimination With Backward Substitution**. Consider the linear system

$$\begin{cases} E_1: & x_1 & - & x_2 & + & 2x_3 & - & x_4 & = & -8 \\ E_2: & 2x_1 & - & 2x_2 & + & 3x_3 & - & 3x_4 & = & -20 \\ E_3: & x_1 & + & x_2 & + & x_3 & & & = & -8 \\ E_4: & x_1 & - & x_2 & + & 4x_3 & + & 3x_4 & = & -8 \end{cases}$$

The augmented matrix is:

$$\begin{bmatrix} 1 & -1 & 2 & -1 & | & -8 \\ 2 & -2 & 3 & -3 & | & -20 \\ 1 & 1 & 1 & 0 & | & -8 \\ 1 & -1 & 4 & 3 & | & -8 \end{bmatrix}$$

and performing the row operations, using a_{11} as the pivot element, with $m_{21} = a_{21}/a_{11} = 2$, $m_{31} = a_{31}/a_{11} = 1$, and $m_{41} = a_{41}/a_{11} = 1$:

$$E_2 - m_{21}E_1 \to E_2, \quad E_3 - m_{31}E_1 \to E_3, \quad \text{and} \quad E_4 - m_{41}E_1 \to E_4$$

gives

$$\begin{bmatrix} 1 & -1 & 2 & -1 & | & -8 \\ 0 & 0 & -1 & 1 & | & -4 \\ 0 & 2 & -1 & -1 & | & 6 \\ 0 & 0 & 2 & 4 & | & 12 \end{bmatrix}$$

Now we need to interchange the second row and third rows ($E_2 \leftrightarrow E3$), we obtain

$$\begin{bmatrix} 1 & -1 & 2 & -1 & | & -8 \\ 0 & 2 & -1 & -1 & | & 6 \\ 0 & 0 & -1 & 1 & | & -4 \\ 0 & 0 & 2 & 4 & | & 12 \end{bmatrix}$$

By choosing a_{33} as the pivot element with $m_{43} = a_{43}/a_{33} = -2$ and $E_4 - m_{43}E_3 \to E_4$, we obtain:

$$\begin{bmatrix} 1 & -1 & 2 & -1 & | & -8 \\ 0 & 2 & -1 & -1 & | & 6 \\ 0 & 0 & -1 & 1 & | & -4 \\ 0 & 0 & 0 & 2 & | & 4 \end{bmatrix}$$

Finally, the backward substitution is applied:

$$\begin{aligned} x_4 &= \frac{4}{2} = 2, \\ x_3 &= \frac{[-4 - (-1)x_4]}{-1} = 2, \\ x_2 &= \frac{[6 - x_4 - (-1)x_3]}{2} = 3, \\ x_1 &= \frac{[-8 - (-1)x_4 - 2x_3 - (-1)x2]}{1} = -7. \end{aligned}$$

♣ **Gaussian Elimination With Partial Pivoting**. The linear system

$$\begin{cases} E_1: & 0.003000x_1 & + & 59.140000x_2 & = & 59.170000 \\ E_2: & 5.291000x_1 & - & 6.130000x_2 & = & 46.780000 \end{cases}$$

has the exact solution $x_1 = 10.00$ and $x_2 = 1.00$. To illustrate the difficulties of roundoff error, Gaussian elimination will be performed on this system using four-digit rounding arithmetic.

The first pivot element $a_{11} = 0.003000$ is small and

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{5.291000}{0.003000} = 1763.666666.$$

Performing

$$E_2 - m_{21}E_1 \to E_1$$

and the appropriate rounding gives

$$\begin{cases} 0.003000\,x_1 & + & 59.140000\,x_2 & = & 59.170000 \\ & - & 104300x_2 & = & -104400 \end{cases}$$

instead of the precise values

$$\begin{cases} 0.003000\,x_1 & + & 59.140000\,x_2 & = & 59.170000 \\ & - & 104309.376666\,x_2 & = & -104409.376666 \end{cases}$$

The disparity in the magnitude of $m_{21}a_{13}$ and $a23$ has introduced roundoff error, but the roundoff has not yet been propagated. Backward substitution yields

$$x_2 \approx 1.001,$$

which is a close approximation to the actual value, $x_2 = 1.000$. However, because of the small pivot $a_1 1 = 0.003000$,

$$x_1 \approx \frac{59.17 - (59.14)(1.001)}{0.003000} = -10.000$$

contains the small error of 0.001 multiplied by

$$\frac{59.14}{0.003000}.$$

This ruin the approximation to the actual value $x_1 = 10.000$.

The partial pivoting method consists of finding

$$\max\{|a_{11}|,\ |a_{21}|\} = \max\{|0.003000|,\ |5.291|\} = |5.291 = |a_{21}|.$$

Then interchanging the first and second rows ($E_1 \leftrightarrow E_2$),

$$\begin{cases} E_1: & 5.291000x_1 & - & 6.130000x_2 & = & 46.780000 \\ E_2: & 0.003000x_1 & + & 59.140000x_2 & = & 59.170000 \\ \end{cases}$$

We have

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{0.003000}{5.291000} = 0.0005670,$$

and the operation

$$E_2 - m_{21}E_1 \to E_1$$

reduces the system to

$$\begin{cases} 5.291\,x_1 & - & 6.130\,x_2 & = & 46.78, \\ & & 59.14x_2 & \approx & 59.14 \end{cases}$$

The four-digit answer resulting from the backward substitution are the correct values:

$$x_1 = 10.000 \quad \text{and} \quad x_2 = 1.000\,.$$

♣ **Gaussian Elimination With Scaled Partial Pivoting**. The linear system

$$\begin{cases} E_1: & 30.00x_1 & + & 591400.00x_2 & = & 591700.00 \\ E_2: & 5.291000x_1 & - & 6.130000x_2 & = & 46.780000 \end{cases}$$

is the same as the previous system and has the exact solution $x_1 = 10.00$ and $x_2 = 1.00$. Using partial scaling with

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{5.291}{30.000} = 0.1764,$$

and $E_2 - m_{21}E_1 \to E_2$ leads to the system

$$\begin{cases} 30.00\,x_1 & + & 591400.00\,x_2 & = & 591700.00 \\ & - & 104300x_2 & \approx & -104400 \end{cases}$$

which has the same inaccurate solutions $x_2 \approx 1.001$ and $x_1 \approx -10.00\,$.

Scaled partial pivoting, also called scaled-column pivoting, is designed for such a system. The first step in this procedure is to define for each row a scale factors_i by

$$s_i = \max\{|a_{ij}| \ : \ j = 1, 2, 3 \ldots, n \ \}.$$

If for some i we have $s_i = 0$, then the system has no unique solution since all the entries in the row are zero. Assuming that this is not the case, then choose the smallest k such that

$$\max\{\frac{|a_{j1}|}{s_j} \ : \ j = 1, 2, 3 \ldots, n \ \} = \frac{|a_{k1}|}{s_k}$$

and performing $E_1 \leftrightarrow E_k$. This ensures that the largest element in each row has a relative magnitude 1 before the comparison for row interchange is performed. Applying scaled-partial pivoting to the system gives:

$$s_1 = \max\{|a_{11}|,\ |a_{12}|\} = \max\{|30.00|,\ |591400|\} = 591400$$

$$s_2 = \max\{|a_{21}|,\ |a_{22}|\} = \max\{|5.291|,\ |-6.130|\} = 6.130\,.$$

Consequently,

$$\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 0.5073 \times 10^{-4}, \qquad \frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631,$$

and the interchanging $E_1 \leftrightarrow E_2$ is made.

Applying Gaussian elimination to the new system

$$\begin{cases} E_1: & 5.291000x_1 & - & 6.130000x_2 & = & 46.780000 \\ E_2: & 30.00x_1 & + & 591400.00x_2 & = & 591700.00 \end{cases}$$

produces the correct results: $x_1 = 10.00$ and $x_2 = 1.00\,.$

Scaled partial pivoting adds a total of

$$n(n-1) + \sum_{k=1}^{n-1} k = n(n-1) + \frac{n(n-1)}{2} = \frac{3}{2}n(n-1)$$

comparisons and

$$\sum_{k=1}^{n} k = \frac{n(n-1)}{2} - 1$$

divisions to the Gaussian elimination procedure. The time to perform a comparison is about the same as addition/subtraction; hence scaled partial pivoting does not add significantly to the computational time required to solve a system for large values of n.

To emphasize the importance of choosing the scale factor only once, consider the amount of additional that would be required if the procedure were modified so that the new scale factors were determined each time a row interchange decision was to be made (Complete or maximal pivoting). In this case the total additional time required to incorporate complete pivoting into Gaussian elimination is

$$\sum_{k=1}^{n} (k^2 - 1) = \frac{n(n-1)(2n+5)}{6}$$

comparisons, plus many divisions.