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$\pi \quad \mathcal{P}\mathcal{I} \quad \mathcal{I}\mathcal{S} \quad \mathcal{I}\mathcal{R}\mathcal{R}\mathcal{A}\mathcal{T}\mathcal{I}\mathcal{O}\mathcal{N}\mathcal{A}\mathcal{L} \quad \pi$
$\pi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \pi$
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First we define the function

$$f(x) = \frac{x^n(1-x^n)}{n!} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{n+k}\,.$$

It is not difficult to show that

$$f^{(r)}(0) = 0 \quad \text{for} \quad r = 0, 1, 2, \ldots, n-1 \quad \text{and} \quad r > 2n\,.$$

For $m = 0, 1, 2, \ldots, n$, we have

$$f^{(n+m)}(x) = \frac{1}{n!}\sum_{k=m}^{n}\binom{n}{k}\frac{(k+n)!}{(k-m)!}(-1)^k x^{n+k}\,.$$

Thus for any $s \in \mathbb{N}$, $f^{(s)}(0)$ is an integer; and since $f(1-x) = f(x)$, the same is also true for $f^{(s)}(1)$. It is clear that if $0 < x < 1$, then

$$0 < f(x) < \frac{1}{n!}\,. \tag{1}$$

It is sufficient to prove that $\pi^2$ is irrational; so we assume the contrary, that $\pi^2 = \dfrac{a}{b}$ for some positive integer $a$ and $b$.

Now for any positive integer $n$, we define

$$F_n(x) = b^n\left[\pi^{2n}f(x) - \pi^{2n-2}f^{(2)}(x) + \pi^{2n-4}f^{(4)}(x) - \cdots\ \cdots + (-1)^n f^{(2n)}(x)\right]\,. \tag{2}$$

Note that $F_n(0)$ and $F_n(1)$ are both integers. A calculation based on (2) shows that

$$\begin{aligned}\frac{d}{dx}\left[F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right] &= \left[F_n''(x) + \pi^2 F_n(x)\right]\sin\pi x \\ &= b^n\pi^{2n+2}f(x)\sin\pi x \\ &= \pi^2 a^n f(x)\sin\pi x\,;\end{aligned}$$

so $$\int_0^1 \pi a^n f(x)\sin\pi x\ dx = \left[\frac{F_n'(x)\sin\pi x}{\pi} - \pi F_n(x)\cos\pi x\Big|_0^1\right] = F_n(1) + F_n(0)\,,$$

which is an integer. But (1) implies that for any integer $n$,

$$0 < \int_0^1 \pi a^n f(x)\sin\pi x\ dx < \frac{\pi a^n}{n!}.$$

For a large $n$, $\dfrac{\pi a^n}{n!} < 1$. The contradiction then follows from the fact that

$$\int_0^1 \pi a^n f(x)\sin\pi x\ dx = F_n(1) + F_n(0)$$

is a positive integer. ■