Inequalities Concerning Eigenvalues

In all that follows, we will denote by $\theta$ the zero column vector and the identity matrix by $I$. Let $A = (a_{ij})$ be an $n \times n$ real or complex matrix; the set of eigenvalues of $A$

$$\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

is called the spectrum of $A$. An eigenvalue with the largest modulus is called a maximal eigenvalue. The spectral radius of $A$ denoted by $\rho(A)$ is the modulus of a maximal eigenvalue. A matrix norm is defined as

$$||A|| = \{\max ||Av||; ||v|| = 1\}.$$

For $i = 1, 2, \ldots, n$, define

$$R_i(A) = \sum_{j=1}^{n} |a_{ij}| \quad \text{and} \quad r_i(A) = R_i(A) - |a_{i,i}|.$$

The row norm of $A$ is defined as follows:

$$||A||_{\infty} = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} = \max_{1 \leq i \leq n} R_i(A).$$

An orthogonal matrix $Q$ (respectively unitary matrix $U$) is a matrix satisfying $Q^tQ = I$ (respectively $U^*U = I$).

We need the following two lemmas:

Lemma.

$$\det (A) = \prod_{i=1}^{n} \lambda_i \quad \text{and} \quad \text{trace} (A) = \sum_{i=1}^{n} \lambda_i.$$  

Note that similar matrices have the same determinant and trace.

Schur’s lemma. $A$ is unitarily similar to an upper triangular matrix $T = (t_{ij})$, i.e., $T = U^*AU$ for some unitary matrix $U$.

Now we can present some important inequalities concerning eigenvalues.

Theorem. For any matrix norm $||.||$, we have $\rho(A) < ||A||$.

Proof. Suppose $Au = \lambda u$, where $u$ is a unit vector (i.e., $||u|| = 1$). Then we have

$$|\lambda| = |\lambda||u|| = ||\lambda u|| = ||Au|| \leq \{\max ||Av||; ||v|| = 1\} = ||A||.$$

Thus $\rho(A) \leq ||A||$. 

Levy-Deslanque theorem. If the matrix $A$ is strictly diagonally dominant, that is

$$|a_{ii}| > r_i(A) \quad \text{for all } i = 1, 2, \ldots, n.$$

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Then $A$ is invertible.

**Proof.** Suppose $\det(A) = 0$, then for some nonzero vector $u = (u_1, u_2, \ldots, u_n)^t$, $Au = \theta$. Now let $k$ be the index where

$$u_k \geq u_i \quad \text{for all} \quad i = 1, 2, \ldots, n.$$ 

Then

$$|a_{kk}| \|u_k\| = \left| -\sum_{j \neq k} a_{kj} u_j \right| \leq \sum_{j \neq r} |a_{kj}| |u_j| \leq |u_k| r_i(A).$$

which contradiction with $|a_{kk}| > r_k(A)$. \hfill $\blacksquare$

A generalization of Levy-Deslanque theorem is presented without a proof.

**Ovals of Cassini.** If

$$|a_{ii}| |a_{jj}| > r_i(A) r_j(A) \quad (i = 1, 2, \ldots, n \text{ and } i \neq j)$$

then $A$ is invertible.

**Gershgorin’s Disks theorem.** The eigenvalues of $A$ lie in the union of the disks $D_i(a_{ii}, r_i(A))$, centered at $a_{ii}$ with the radius $r_i(A)$.

**Proof.** Let $\lambda_k$ be an eigenvalue of $A$, then $\det(A - \lambda_k I) = 0$. By the Levy-Deslanque theorem, we conclude that $\lambda_k - a_{ii} < r_i(A)$ for at least one $i$. \hfill $\blacksquare$

The fact that $\sigma(A) = \sigma(A^t)$, we can obtain similar results by using columns of $A$ instead of its rows.

**Schur’s Inequalities.**

$$\sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i,j=1}^n |a_{i,j}|^2.$$ 

**Proof.** According to Schur’s lemma, $T = U^* AU$ for some upper triangular matrix $T = (t_{ij})$ and unitary matrix $U = (u_{ij})$. Thus $T^* = U^* A^* U$ and $TT^* = (U^* AU)(U^* A^* U) = U^* AA^* U$. The facts that $\text{trace}(AA^*) = \text{trace}(TT^*)$ and $\text{trace}(AA^*) = \sum_{i,j=1}^n |a_{ij}|^2$ imply that

$$\sum_{i,j=1}^n |a_{ij}|^2 = \text{trace}(AA^*) = \text{trace}(TT^*) = \sum_{i=1}^n |\lambda_i|^2 + \sum_{i,j=1}^n |t_{ij}|^2.$$ 

Hence the desired conclusion. \hfill $\blacksquare$