Integration by Parts

Consider the following integral:

$$I = \int u(x)v(x)\,dx.$$

By differentiating $u(x)$ and integrating $v(x)$ as many times as needed, we may find I by using the following chart.

$$\boxed{\begin{array}{c} + \\ \\ - \\ \\ + \\ \vdots \\ (-1)^{k-1} \\ \\ (-1)^k \end{array}
\left\{\begin{array}{ccc}
u_0(x)=u(x) & & v_0(x)=v(x) \\
 & \searrow & \\
u_1(x)=u'(x) & & v_1(x)=\int v_1(x)dx \\
 & \searrow & \\
u_2(x)=u''(x) & & v_2(x)=\int v_2(x)dx \\
\vdots & \ddots & \vdots \\
u_{k-1}(x)=u^{(k-1)}(x) & & v_{k-1}(x)=\int v_{k-2}(x)dx \\
 & \searrow & \\
u_k(x)=u^{(k)}(x) & \xrightarrow{\int} & v_k(x)=\int v_{k-1}(x)dx
\end{array}\right.}$$

Thus

$$I = \int u(x)v(x)\,dx = \sum_{i=0}^{k-1}(-1)^k u_i(x)v_{i+1}(x) + (-1)^k \int u_k(x)v_k(x)\,dx.$$

♣ **Examples**

$$I = \int x^3 e^{2x} dx$$

$$\begin{array}{c} + \\ \\ - \\ \\ + \\ \\ - \\ \\ + \end{array}
\left\{\begin{array}{ccc}
u_0(x)=x^3 & & v_0(x)=e^{2x} \\
 & \searrow & \\
u_1(x)=3x^2 & & v_1(x)=\frac{1}{2}e^{2x} \\
 & \searrow & \\
u_2(x)=6x & & v_2(x)=\frac{1}{4}e^{2x} \\
 & \searrow & \\
u_3(x)=6 & & v_3(x)=\frac{1}{8}e^{2x} \\
 & \searrow & \\
u_4(x)=0 & \xrightarrow{\int} & v_4(x)=\frac{1}{16}e^{2x}
\end{array}\right.$$

$$I = e^{2x}[\tfrac{x^3}{2} - \tfrac{3x^2}{4} + \tfrac{3x}{4} - \tfrac{3}{8}] + C$$

$$J = \int e^{2x}\sin x dx$$

$$\begin{array}{c} + \\ \\ - \\ \\ + \end{array}
\left\{\begin{array}{ccc}
u_0(x)=e^{2x} & & v_0(x)=\sin x \\
 & \searrow & \\
u_1(x)=2e^{2x} & & v_1(x)=-\cos x \\
 & \searrow & \\
u_2(x)=4e^{2x} & \xrightarrow{\int} & v_2(x)=-\sin x
\end{array}\right.$$

$$J = e^{2x}[-\cos x + 2\sin x] - 4J$$

$$J = \tfrac{1}{5}[-\cos x + 2\sin x] + C$$

The following integrals involve integration by parts, where $v_0(x) = 1$:

$$\int \ln x\ dx \quad \int \sin^{-1} x\ dx \quad \int \cos^{-1} x\ dx \quad \int \tan^{-1} x\ dx \quad \int \cot^{-1} x\ dx \quad \int \sec^{-1} x\ dx \quad \int \csc^{-1} x\ dx$$