Jordan Canonical Form

Jordan normal form or Jordan canonical form (named in honor of Camille Jordan) shows that by changing the basis, a given square matrix M can be transformed into a certain normal form

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & J_p \end{pmatrix},$$

where each block J_i is a square matrix of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}.$$

Note that $\lambda_i' s$ are the repeated eigenvalues of M.

1. Given an eigenvalue λ_i of an $n \times n$ matrix M, its *geometric multiplicity* is the dimension of $Ker(M - \lambda_i I_n)$, and it is the number of Jordan blocks corresponding to λ_i.
2. The sum of the sizes of all Jordan blocks corresponding to an eigenvalue λ_i is its algebraic multiplicity.
3. M is *diagonalizable* if and only if, for any eigenvalue λ of M, its *geometric* and *algebraic multiplicities* coincide. A non diagonalizable matrix is sometimes called a *defective matrix.*

To obtain a Jordan normal form of an $n \times n$ diagonalizable matrix M, we use n linearly independent eigenvectors to construct a matrix P, where the diagonal matrix $P^{-1}MP$ will be the Jordan normal form.

Next we shall discuss the case of defective matrices.

♣ **Repeated Eigenvalues**. Consider the following 4×4 matrices:

$$A = \begin{pmatrix} -30 & -12 & 19 & 27 \\ -10 & 1 & 5 & 8 \\ -6 & -2 & 7 & 5 \\ -43 & -15 & 24 & 38 \end{pmatrix}, \qquad B = \begin{pmatrix} -12 & -6 & 9 & 13 \\ -1 & 4 & 0 & 1 \\ -6 & -2 & 7 & 5 \\ -16 & -6 & 9 & 17 \end{pmatrix},$$

$$C = \begin{pmatrix} -25 & -10 & 16 & 23 \\ -20 & -3 & 11 & 16 \\ -11 & -4 & 10 & 9 \\ -38 & -13 & 21 & 34 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} -7 & -4 & 6 & 9 \\ -11 & 0 & 6 & 9 \\ -11 & -4 & 10 & 9 \\ -11 & -4 & 6 & 13 \end{pmatrix}$$

with the characteristic polynomials:

$$K_A(\lambda) = K_B(\lambda) = K_C(\lambda) = K_D(\lambda) = \lambda^4 - 16\lambda^3 + 96\lambda^2 - 256\lambda + 256 = (\lambda - 4)^4.$$

Thus $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 4$.

Case 1. The geometric multiplicity of the matrix A is one, so there is only one Jordan block.

The rank of the matrix $\widehat{A} = A - 4I_4 = \begin{pmatrix} -34 & -12 & 19 & 27 \\ -10 & -3 & 5 & 8 \\ -6 & -2 & 3 & 5 \\ -43 & -15 & 24 & 34 \end{pmatrix}$ is 3, so the vector $u(A) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ which is the solution of the homogeneous system $\widehat{A}u = \theta$ is the only linearly independent eigenvector of A.
The vectors

$$v(A) = \begin{pmatrix} -1 \\ -2 \\ -3 \\ 0 \end{pmatrix}, \quad w(A) = \begin{pmatrix} 3 \\ 9 \\ 11 \\ 0 \end{pmatrix}, \quad \text{and} \quad z(A) = \begin{pmatrix} 0 \\ -16 \\ -27 \\ 12 \end{pmatrix}$$

which are called the generalized eigenvectors of A are obtain by solving

$$\widehat{A}v = u(A), \quad \widehat{A}w = v(A), \quad \text{and} \quad \widehat{A}z = w(A) \text{ respectively.}$$

By constructing the matrix

$$P = [u(A)\ v(A)\ w(A)\ z(A)] = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 1 & -2 & 9 & -16 \\ 1 & -3 & 11 & -27 \\ 1 & 0 & 0 & 12 \end{pmatrix} \quad \text{with} \quad P^{-1} = \begin{pmatrix} -60 & -24 & 36 & 49 \\ -43 & -15 & 24 & 34 \\ 6 & 3 & -4 & -5 \\ 5 & 2 & -3 & -4 \end{pmatrix},$$

we obtain the Jordan canonical form of A as follows:

$$J(A) = P^{-1}AP = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Case 2a. The geometric multiplicity of the matrix B is two, so there are two blocks in the Jordan normal matrix. Since the rank of the matrix $\widehat{B} = B - 4I_4 = \begin{pmatrix} -16 & -6 & 9 & 13 \\ -1 & 0 & 0 & 1 \\ -6 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \end{pmatrix}$ is 2, the vectors

$$u_1(B) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad u_2(B) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

which are the solution of the homogeneous system $\widehat{B}u = \theta$ are linearly independent eigenvectors of B. To obtain the Jordan normal form of B, we need to obtain two generalized eigenvectors of B by solving the systems $\widehat{B}v = u_1(B)$ and $\widehat{B}v = u_2(B)$. The generalized eigenvectors are

$$v_1(B) = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2(B) = \begin{pmatrix} -2 \\ 0 \\ -5 \\ 1 \end{pmatrix}.$$

The Jordan normal form of B is obtain from the matrix

$$Q = [u_1(B)\ v_1(B)\ u_2(B)\ v_2(B)] = \begin{pmatrix} 1 & -1 & 0 & -2 \\ 1 & 1 & 3 & 0 \\ 1 & -1 & 2 & -5 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad Q^{-1} = \begin{pmatrix} -5 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \\ 7 & 3 & -4 & -6 \\ 5 & 2 & -3 & -4 \end{pmatrix}.$$

We have

$$J(B) = Q^{-1}BQ = \left(\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right).$$

Case 2b. The geometric multiplicity of the matrix C is also two, so there are two blocks in the Jordan normal form. Since the rank of the matrix

$\widehat{C} = C - 4I_4 = \begin{pmatrix} -29 & -10 & 16 & 23 \\ -20 & -7 & 11 & 16 \\ -11 & -4 & 6 & 9 \\ -38- & 13 & 21 & 30 \end{pmatrix}$ is 2, the vectors $u_1(C) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2(C) = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 2 \end{pmatrix}$

are linearly independent eigenvectors of C. By solving the system $\widehat{C}v = u_1(C)$, we obtain two linearly independent generalized eigenvectors

$$v_{11}(C) = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v_{12}(B) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

but the vectors $u_1(C)$, $u_2(C)$, $v_{11}(C)$ and $v_{12}(C)$ are linearly dependent; so we should use one of the two options:

1. Remove one of the generalized eigenvectors and solve the system $\widehat{C}v = u_2(C)$.
2. Solve either $\widehat{C}w = v_{11}(C)$ or $\widehat{C}w = v_{12}(C)$.

The system $\widehat{C}v = u_2(C)$ is inconsistent, so we use the second option. The system $\widehat{C}w = v_{11}(C)$ is also inconsistent, but $\widehat{C}w = v_{12}(C)$ has the vector $w(C) = \begin{pmatrix} -2 \\ 8 \\ 0 \\ 1 \end{pmatrix}$ as a solution. Now by choosing $u_1(C)$, $u_2(C)$, $v_{12}(C)$ and $w(C)$, we may construct the matrix

$$R = [u_1(C)\ v_{12}(C)\ w(C)\ u_2(C)] = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 8 & 3 \\ 1 & -1 & 0 & -1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \text{ with } R^{-1} = \begin{pmatrix} 3 & 1 & -1 & -2 \\ -20 & -7 & 11 & 16 \\ -9 & -3 & 5 & 7 \\ 23 & 8 & -13 & -18 \end{pmatrix}.$$

The Jordan canonical form of C is as follows:

$$J(C) = R^{-1}CR = \left(\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ \hline 0 & 0 & 0 & 4 \end{array}\right).$$

Case 3. The rank of the matrix $\widehat{D} = D - 4I_4 = \begin{pmatrix} -11 & -4 & 6 & 9 \\ -11 & -4 & 6 & 9 \\ -11 & -4 & 6 & 9 \\ -11 & -4 & 6 & 9 \end{pmatrix}$ is one, so there are three blocks in the Jordan normal form. The three linearly independent eigenvectors of D are as follows:

$$u_1(D) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ u_2(D) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix},\ \text{and } u_3(D) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -2 \end{pmatrix}$$

The generalized eigenvector $v_2(D)\begin{pmatrix}0\\-1\\1\\-1\end{pmatrix}$ is a solution of the system $\widehat{D}v = u_1(D)$.

Note that the eigenvectors $\begin{pmatrix}0\\3\\2\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\3\\-2\end{pmatrix}$, and $\begin{pmatrix}1\\4\\0\\3\end{pmatrix}$ are linearly independent but do not produce any generalized eigenvector since all the rows of $\widehat{D}$ are identical but components of these three vectors are not all equal. The matrix

$$S = [u_1(D)\ v_1(D)\ u_2(D)\ u_3(D)] = \begin{pmatrix}1&0&0&0\\1&-1&3&0\\1&1&2&3\\1&-1&0&-2\end{pmatrix} \text{ with } S^{-1} = \begin{pmatrix}1&0&0&0\\-11&-4&6&9\\-4&-1&2&3\\6&2&-3&-5\end{pmatrix}$$

will produce the Jordan normal form as follows:

$$J(D) = S^{-1}DS = \left(\begin{array}{cc|c|c}4&1&0&0\\0&4&0&0\\\hline 0&0&4&0\\\hline 0&0&0&4\end{array}\right).$$

♣ **System of Linear Differential Equations**. Consider the system

$$X'(t) = AX(t) \quad \text{with} \quad T^{-1}AT = J(A).$$

Let $X(t) = TY(t)$, then $X'(t) = TY(t)$, $T^{-1}X(t) = Y(t)$, and $T^{-1}X'(t) = Y('t)$. By solving the system $Y'(t) = J(A)Y(t)$, we may obtain the solution of the original system as follows:

$$X'(t) = TY'(t) = T\,[J(A)Y(t)] = T\left[T^{-1}ATY(t)\right] = \left[TT^{-1}\right]ATY(t) = \left[TT^{-1}\right]A\,[TY(t)] = AX(t).$$