First Order Differential Equations

The k-th order derivative of the function $y(x)$ is denoted by $D_x^k y$ or simply $D^k y$. Thus

$$D^k y = D_x^k y = \frac{d^k y}{dx^k} \quad and \quad D_x^0 y = Iy = y.$$

A general n-th order, ordinary differential equation is represented by

$$F(x, y, Dy, \cdots, D^n y) = 0;$$

so an ordinary differential equation is an equation (E) which contains terms such as $D^k y$. The highest power of D in (E) is called the order of the equation.

The equation $F(x, D)y = R(x)$, where

$$F(x, D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_n(x)I,$$

is said to be *linear of order n.* When $R(x) = 0$, then the linear differential equation is called *homogeneous.*

If a solution of $F(x, y, D) = 0$ can be expressed as $y = f(x)$ (i.e., y is a function of x), then this solution is called an *explicit solution.* If we obtain $f(x, y) = 0$ as a solution of our differential equation, then we say that only an *Implicit solution* has been found.

First Order Differential Equations

A first order differential equation may be expressed as follows:

$$\frac{dy}{dx} = f(x, y).$$

The problem

$$\begin{cases} \text{Solve:} & \frac{dy}{dx} = f(x, y) \\ \text{Subject to:} & y(x_0) = y_0 \end{cases} \tag{1}$$

is called an *initial-value problem.* The first equation gives the slope of the curve y at any point x, and the second equation specifies one particular value of the function $y(x)$.

Existence. Will every initial-value problem have a solution? No, some assumptions must be made about $f(x, y)$, and even then we can only expect the solution to exist in a neighborhood of $x = x_0$. As an example of what could happen, consider

$$\begin{cases} \dfrac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$$

The solution curve starts at $x = 0$ with slope one; that is, $y'(0) = 1$. Since the slope is positive, $y(x)$ is *increasing* near $x = 0$. Therefore, the expression $1 + y^2$ is also increasing. Hence, y' is increasing. Since y and y' are both increasing and are related by the equation $y' = 1 + x^2$, we can expect that at some finite value of x there will be no solution; that is,

$y(x) = +\infty$. As a matter of fact, this occurs at $x = \pi/2$ because the *analytic solution* of the initial-value problem is $y(x) = \tan x$.

Theorem 1. *If $f(x,y)$ is continuous in a rectangle R centered at (x_0, y_0), say*

$$R = \{(x,y) : |x - x_0| \leq \alpha, \quad |y - y_0| \leq \beta\} \tag{2}$$

then the initial-value problem (1) *has a solution $y(x)$ for $|x - x_0| \leq min\,(\alpha, \beta/M)$, where M is the maximum of $|f(x,y)|$ in the rectangle R.*

Uniqueness. It can happen, even if $F(x,y)$ is continuous, that the initial-value problem does not have a unique solution. A simple example of this phenomenon is given by the problem

$$\begin{cases} \dfrac{dy}{dx} = y^{2/3} \\ y(0) = 0 \end{cases}$$

It is obvious that the zero function, $y(x) \equiv 0$, is a solution of this problem. Another solution is the function

$$y(x) = \frac{1}{27}x^3$$

To prove that the initial-value problem (1) has a *unique solution* in a neighborhood of $x = x_0$, it is necessary to assume somewhat more about $f(x,y)$. Here are the usual theorems on this.

Theorem 2. *If $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous in a rectangle R defined by* (2)*, then the initial-value problem* (1) *has a unique solution in the interval $|x - x_0| \leq min\,(\alpha, \beta/M)$.*

Theorem 3. *If $f(x,y)$ is continuous in the strip*

$$a \leq x \leq b, \quad -\infty < y < \infty$$

and satisfies there an inequality

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| (3)$$

then the initial-value problem (1) *has a unique solution in the interval $[a, b]$.*

The Inequality (3) is called a *Lipschitz condition* in the second variable. We see immediately that this condition is *stronger* than continuity.

Let $f(u)$ be a function where $u \in \mathbb{R}^n$. Then we say that $f(u)$ is *homogeneous of degree k,* if $f(\lambda u) = \lambda^k f(u)$, for a suitable real λ.

♠ **Note.** Do not confound homogeneous equations with homogeneous functions.

There are several classes of differential equations of order one. We shall explain how to classify and solve some of these classes.

♣ **Separation of Variables**. If $M(x,y) = M(x)$ and $N(x,y) = N(y)$, then (E) may be written as $M(x)dx = -N(y)dy$. By integrating both sides of the equality we solve the equation.

Equation:	$2xydx - (x^2+1)dy = 0$
Step 1.	$[2x/(x^2+1)]dx = dy/y$
Step 2.	$\int[2x/(x^2+1)]dx = \int dy/y$
Implicit Solution	$\ln(x^2+1) = \ln\|y/C\|$
Explicit Solution	$y = C(x^2+1)$

♣ **Homgeneous Coefficients**. If $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the same degree. Then by using a substitution we may solve the equation by the method of separation of variables.

Algorithm. Set $y = ux$ (or $x = vy$) in (E), then $dy = xdu + udx$ (or $dx = ydv + vdy$). We obtain

$$\widehat{M}(x,u)dx + \widehat{N}(x,u)du = 0 \text{ or } \widehat{M}(v,y)dv + \widehat{N}(v,y)dy = 0$$

which can be solved by using separation of variables.

Equation:	$(x^2 + 2y^2)dx - xydy = 0$
Step 1.	$M(\lambda x, \lambda y) = \lambda^2 M(x,y) \quad N(\lambda x, \lambda y) = \lambda^2 N(x,y)$
Step 2.	$y = xu \quad dy = xdu + udx$
Step 3.	$dx/x = \int [u/(1+u^2)]du$
Implicit Solution	$Cx^4 - x^2 = y^2$

If the point (h,k) is a solution to the linear system

$$\begin{cases} a_1 x + b_1 y + c_1 = 0 \\ a_2 x + b_2 y + c_2 = 0. \end{cases}$$

Then by setting $x = u + h$ and $y = v + k$ in (E), we obtain the equation

$$(a_1 u + b_v)du + (a_2 u + b_2 v)dv = 0$$

which has homogeneous coefficients of degree one.

If $(a_1 x + b_1 y + c_1) = s(a_2 x + b_2 y + c_2) + r$, then by substituting u for $(a_1 x + b_1 y + c_1)$ and eliminating x or y, we may solve the equation by separation of variables.

♣ **Eact Equation**. The equation (E) is said to be exact, if $M_y(x,y) = N_x(x,y)$. Since $F_{xy}(x,y) = F_{yx}(x,y)$ for any smooth function $F(x,y)$, we conclude that there exists a constant function $C = F(x,y)$ such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$ with

$$0 = d\,C = d\,F(x,y) = M(x,y)dx + N(x,y)dy.$$

Algorithm. Let

$$C = F(x,y) = \int M(x,y)dx + T(y) \tag{1}$$

$$C = F(x,y) = \int N(x,y)dy + S(x). \tag{2}$$

We select (1) or (2), whichever is simpler and easier to integrate.

By setting $\partial F(x,y)/\partial y = N(x,y)$ (or $\partial F(x,y)/\partial x = M(x,y)$) we obtain $T'(y)$ or $S'(x)$. To find $T(y)$ or $S(x)$ we just integrate $T'(y)$ or $S'(x)$. The solution of (E) is then obtained from (1) or (2).

Equation:	$(x+2y)dx + (2x+y)dy = 0$
Step 1.	$M_y(x,y) = N_x(x,y) = 2$
Step 2.	$C = F = \int (x+2y)dx + T(y) \quad F = (\frac{x^2}{2}) + 2xy + T(y)$
Step 3.	$2x + T'(y) = (2x+y) \quad T(y) = y^2/2$
Implicit Solution	$C = \frac{x^2}{2} + 2xy + y^2$

♢ **Example 1.** If

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} \qquad \text{or} \qquad -\frac{[M_y(x,y) - N_x(x,y)]}{M(x,y)}$$

is a function of x (or y). Then the integrating factor is

$$u(x) = exp\left(\int \frac{M_y(x,y) - N_x(x,y)}{N}dx\right) \qquad \text{or} \qquad v(y) = \exp\left(-\int \frac{M_y(x,y) - N_x(x,y)}{N}dy\right).$$

♢ **Example 2.** If $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the same degree and $xM(x,y) + yN(x,y) \neq 0$, then

$$u(x,y) = \frac{1}{xM(x,y) + yN(x,y)}$$

is an integrating factor for the equation.

♣ **Linear Equation**. The first order linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

can be solved by defining first the function

$$v(x) = \exp\left(\int p(x)dx\right)$$

and then finding

$$R(x) = \int v(x)Q(x)dx\,.$$

The solution to the equation is $xy = R(X)$.

♡ **Bernouilli's Equation**. The following equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (n \neq 1)$$

is called a Bernouilli's equation. By setting $z = y^{-n+1}$ in the equation, we obtain the linear equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Equation:	$\frac{dy}{dx} + \frac{1}{x}y = xy^2$
Step 1.	$z = y^{-2+1}$
Step 2.	$\frac{dz}{dx} - \frac{1}{x}z = -x$
Step 3.	$v(x) = e^{-\int dx/xdx} = x^{-1}$
Step 4.	$zx^{-1} = \int -x(x^{-1})dx = -\int dx \quad y^{-1}x^{-1} = -x - C$
Explicit Solution	$y = \frac{-1}{x^2+Cx}$

♡ **Ricatti's Equation**. The nonlinear equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

which frequently occurs in physical applications is called Ricatti's equation. Its solution cannot be expressed in terms of elementary function. However when $R(x) = -1$, we can change the equation into a second order linear differential equation by setting $y = z'/z$.

♡ **Clairaut's Equation**. The nonlinear equation

$$y = xy' + f(y')$$

is called Clairaut's equation. By differentiating both sides of the equality with respect to x, we obtain the second order equation

$$[x + f'(y')]\, y'' = 0.$$

One set of solutions called *general solution,* is $y = cx + f(c)$ and is obtained from $y'' = 0$.

If $x + f'(y') = 0$, then we obtain the parametrized curve

$$x = -f'(t), \qquad y = f(t) - tf'(t).$$

This curve is also a solution, called the *singular solution,*

♣ **Solving by Inspection**. The following identities may help you solve some differential equations.

$ydx + xdy = d(xy)$	$\frac{ydx-xdy}{y^2} = d\left(\frac{x}{y}\right) = -d\left(\frac{y}{x}\right)$
$mx^{m-1}y^n dx + nx^m y^{n-1}dy = d(x^m y^n)$	$\frac{mx^{m-1}y^n dx - nx^m y^{n-1}dy}{y^{2n}} = d\left(\frac{x^m}{y^n}\right)$
$\frac{ydx+xdy}{xy} = d(\ln(xy))$	$\frac{ydx-xdy}{xy} = d\left[\ln\left(\frac{x}{y}\right)\right] = -d\left[\ln\left(\frac{y}{x}\right)\right]$
$\frac{ydx+xdy}{1+x^2y^2} = d\left[\arctan(xy)\right]$	$\frac{ydx-xdy}{x^2+y^2} = d\left[\arctan\left(\frac{x}{y}\right)\right] = -d\left[\arctan\left(\frac{y}{x}\right)\right]$

♣ **Picard's Successive Approximations**. Consider the initial-value problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

If we integrate both sides of the differential equation form x_0 to x with respect to x, we obtain the new equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \qquad (E_1)$$

Since $y(x_0) = y_0$, the new equation (E_1) is an alternative way of writing the initial-value problem. Furthermore, if we differentiate both sides of (E_1), we obtain the differential equation

$$y'(x) = f(x, y(x)).$$

We now define a sequence of functions $\{y_n(x)\}$, called *Picard's iterations,* by successive formulas:

Picard's method:

$y' = f(x, y), \qquad y(x_0) = y_0$

$$\begin{aligned}
y_0(x) &= y_0 \\
y_1(x) &= y_0 + \textstyle\int_{x_0}^{x} f\big[t, y_0(t)\big]dt, \\
y_2(x) &= y_0 + \textstyle\int_{x_0}^{x} f\big[t, y_1(t)\big]dt, \\
&\;\;\vdots \qquad\qquad \vdots \\
y_n(x) &= y_0 + \textstyle\int_{x_0}^{x} f\big[t, y_{n-1}(t)\big]dt,
\end{aligned}$$

Equation:

$y_0 = y(x), \qquad y(0) = 1$

$$\begin{aligned}
y_0(x) &= y_0 = 1 \\
y_1(x) &= 1 + \textstyle\int_{x_0}^{x} (1)dt = 1 + x \\
y_2(x) &= 1 + \textstyle\int_{x_0}^{x} (1+t)dt = 1 + x + \frac{x^2}{2}, \\
&\;\;\vdots \qquad\qquad \vdots \\
y_n(x) &= 1 + x + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!}
\end{aligned}$$