Roots of Polynomials

\* Best Neighborhoods for the Roots of Polynomials. \ Let $A = (a_{ij})$ be an $n \times n$ matrix. If $Au = \lambda u$, then $\lambda$ and $u$ are called the eigenvalue and eigenvector of $A$, respectively. The eigenvalues of $A$ are the roots of the characteristic polynomial

$$K_A(\lambda) = \det(\lambda I - A).$$

The eigenvectors are the solutions to the Homogeneous system

$$(\lambda I - A)X = \theta.$$ 

If $A$ is symmetric, i.e., $A^t = A$, then all the eigenvalues of $A$ are real. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A$, then

$$\text{Trace } (A) = \sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} a_{ii} \quad \text{and} \quad \det A = \prod_{k=1}^{n} \lambda_k.$$

Our first theorem is known as the Gerschgorin’s Disks Theorem.

**Theorem 1.** Let $A = (a_{ij})$ be an $n \times n$ matrix. For $j = 1, 2, \ldots, n$, define

$$r_j = \left( \sum_{i=1}^{n} |a_{ij}| \right) - |a_{jj}|.$$

Let $D_j(a_{jj}, r_j)$ be the disk of radius $r_j$ with the center at the point $(0, a_{jj})$ of the complex plane. Then all the eigenvalues of the matrix $A$ is contained within the union of the $D_j$’s. Thus

$$D(A) = \bigcup_{j=1}^{n} D_j$$

contains all the eigenvalues of $A$.

**Remark.** Since $A$ and $A^t$ have the same set of eigenvalues, we may use Theorem 1 for both $A$ and $A^t$ and get the best neighborhood $D(A) \cap D(A^t)$ for the eigenvalues of $A$.

Consider now the polynomial of degree $n$

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.$$ 

The polynomial $P$ is said to be monic, if the leading coefficient $a_0$ equals one. To this monic matrix we associate an $n \times n$ matrix $C_p$, called the Companion Matrix of $P(x)$.

$$C_p = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & \cdots & \cdots & -a_1
\end{pmatrix}.$$ 

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Theorem 2. \(x_0\) is a root of \(p(x)\) if and only if \(x_0\) is an eigenvalue of the matrix \(C_p\).

Corollary. Consider a monic polynomial \(P(x)\) of degree \(n\). Then
(i) all the roots of \(P(x)\) is contained within \(D_r \cap D_c\), where
\[
D_r = \left[ D(0, 1) \cup D \left(-a_{n-1}, \sum_{k=0}^{n-2} |a_k| \right) \right]
\]
and
\[
D_c = [D(0, |a_{n-1}|) \cup D(0, 1 + |a_{n-2}|) \cup \ldots \cup D(0, 1 + |a_2|) \cup D(-a_1, 1)];
\]
(ii) if \(\{x_1, x_2, \ldots, x_n\}\) are the \(n\) roots of \(P(x)\), then
\[
\sum_{k=1}^{n} = -a_1.
\]

Proof. By using \(C_p\), the above theorems, the Remark and the fact that \(\text{Trace}(C_p) = -a_1\); one may readily prove the corollary.

\[\blacksquare\] Rational Roots. Although a real polynomial may have complex roots, but there is a well known theorem concerning the rational roots of polynomial with integer coefficients.

Theorem 3. Let \(P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n\) be a polynomial with integer coefficients. If \(p/q\) is a rational root of \(P(x)\), then \(a_n = pr\) and \(a_0 = qs\).

\[\blacksquare\] Nested Form. Consider the following polynomial of degree \(n\)
\[
P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n.
\]
The following form of \(P(x)\) is called the nested form of \(P(x)\):
\[
P(x) = (((((a_0)x + a_1)x + a_2)x \cdots )x + a_{n-1})x + a_n).
\]
Finally, we present a root finding tool known as Horner’s method or Synthetic division.

\[\blacksquare\] Synthetic Division. Consider the polynomial:
\[
P(x) = 2x^4 - 3x^2 + 3x - 4 = (((2)x + 0)x - 3)x + 3)x - 4).
\]
The following chart shows how to evaluate \(P(a)\) for \(a = -2\).

<table>
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<tr>
<th>(-2)</th>
<th>(-2)</th>
<th>(-4)</th>
<th>(8)</th>
<th>(-10)</th>
<th>(14)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>(-3)</td>
<td>3</td>
<td>(-4)</td>
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<td>(\uparrow)</td>
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</tr>
<tr>
<td>0</td>
<td>(-4)</td>
<td>(8)</td>
<td>(-10)</td>
<td>(14)</td>
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</tr>
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</table>

\(2 \rightarrow -4 \rightarrow 5 \rightarrow -7 \rightarrow 10 = p(-2)\)