

An Efficient Approximation of the Traveling Salesman Polytope Using Lifting Methods

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Abstract

For the Traveling Salesman Polytope on n cities T_n , we construct its approximation $Q_k, k = 1, 2, \dots, n^{1/3}$ using a projection of a polytope whose number of facets is polynomial in n (of degree linear in k). We show that T_n is contained in Q_k for each k , and that the scaling of Q_k by $\frac{k}{n} + O(\frac{1}{n})$ is contained in T_n for each k . We show that certain facets of T_n lie on the boundary of Q_k .

1 Introduction and Results

For many interesting convex bodies X in a vector space V , given a point $x \in V$, the question “is x in X ?” is difficult to answer. This fact has generated work in the direction of finding another set Y which is “close” to X in some way for which the membership question is “easy” to answer. The following results have been the motivation for this paper.

1.1 Polytope Projection and Successive Approximation

Given a convex body X in a vector space V , a natural type of set to use to approximate X is a polytope. Unfortunately, in order to get a “good” approximation, this may require that the polytope have exponentially many facets (exponential in $\dim(X)$). For example, if B_d is the Euclidian ball in \mathbb{R}^d , any polytope containing B_d must have exponentially many facets to have its volume be within a factor c^d of the volume of B_d for any constant c (see, for example, section 13.2 of [7]).

This has led to the idea of approximating convex bodies by *projections* of polytopes, the point being that the projection of a polytope may have many more facets than the original polytope. Ben-Tal and Nemirovski have exploited this fact in the case of a Euclidean Ball. In [2], they proved that for any $\epsilon > 0$, and nonnegative integer n , there exists $N = O(n \log(\epsilon^{-1}))$, a polytope P with no more than N facets, and a linear transformation T such that

$$T(P) \subset B_n \subset (1 + \epsilon)T(P)$$

where $B_n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is the Euclidean unit ball in \mathbb{R}^n . Note that an arbitrary convex X can be approximated by an ellipsoid within a factor of $\dim(X)$ (see, for example, section V.2 of [1]). Thus, for $n = \dim(X)$, this automatically gives a polytope P whose number of facets is polynomial in n and $\log(\epsilon^{-1})$ and a linear transformation T such that

$$T(P) \subset X \subset (1 + \epsilon)nT(P)$$

Sherali and Adams [9], Lovász and Schrijver [6], and Lasserre [4] have also used projections of polytopes. In each of these instances, the authors constructed successive relaxations of a 0-1 polytope (each of which was a projection of another polytope), such that in the n th step, the 0-1 polytope is achieved: $P = K^n \subset K^{n-1} \subset \dots \subset K^1 \subset K$. Metric properties of these sets are not known. For specifics, as well as a comparison of the methods, see [5].

1.2 Approximation of the STSP

The Symmetric Traveling Salesman Polytope (STSP) can be described as follows: recall that a Hamiltonian cycle in the complete graph on n vertices K_n is a cycle which visits every vertex exactly once. To each Hamiltonian cycle in K_n , we can associate its incidence matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if the cycle contains edge } \{i, j\} \\ 0 & \text{if the cycle does not contain edge } \{i, j\} \end{cases}$$

It is called symmetric because there is a similar notion in the case of a digraph (a graph where the edges have an orientation), which is the Asymmetric Traveling Salesman polytope.

Note that each matrix corresponding to a Hamiltonian cycle is a symmetric 0-1 matrix in \mathbb{R}^{n^2} with 0s on the diagonal. Given a particular matrix corresponding to a Hamiltonian cycle, any other such matrix can be achieved from it by simultaneously permuting rows and columns (this corresponds to permuting the labels on the vertices of the graph). The Traveling Salesman Polytope is the convex hull of all adjacency matrices corresponding to Hamiltonian cycles in K_n . Note that the vertices of the STSP are matrices which correspond to cycles. To each cycle we can associate a permutation of the numbers $\{1, 2, \dots, n\}$ beginning with the number 1 where the permutations $(1, m_2, m_3, \dots, m_n)$ and $(1, m_n, m_{n-1}, \dots, m_2)$ are identified. We will use the descriptions of the vertices as matrices, cycles, and permutations interchangeably. With the permutation description of a Hamiltonian cycle, it is not hard to see that there are $\frac{(n-1)!}{2}$ different Hamiltonian cycles in K_n .

The STSP has been studied widely, though a complete description of it via linear inequalities is unknown. It is clearly not full dimensional in \mathbb{R}^{n^2} , being the convex hull of symmetric matrices with 0s on the diagonal. It is not hard to show that its dimension is $\frac{n(n-3)}{2}$. For more information on the STSP and the associated Traveling Salesman Problem, see, for example, Chapter 58 of [8].

Linear optimization over the STSP and the membership question for the STSP are known to be NP-hard.

Let X denote the set of matrices corresponding to Hamiltonian cycles in K_n . Instead of working directly with the STSP, we will be working with its polar. Note that the barycenter of the STSP is the matrix $Z = (z_{ij})$ where

$$z_{ij} = \begin{cases} \frac{2}{n-1} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

Since the STSP is not full dimensional, in order to get a bounded polar we move Z to the origin, which forces the average value to be 0:

$$A = \left\{ \text{linear functions } f : \frac{1}{|X|} \sum_{x \in X} f(x) = 0, \quad f(x) \leq 1 \quad \text{for all } x \in X \right\}$$

Then for pure convenience, we will reflect $A \mapsto -A$ and then shift $f \mapsto f + 1$ to obtain the following description of the dual that will be the one with which we work:

$$Q = \left\{ \begin{array}{l} \text{functions } f : f \text{ is linear, that is } f(x) = \langle c, x \rangle \text{ for some matrix } c \\ \text{non-negative, that is } f(x) \geq 0 \text{ for all } x \in X \\ \text{has average 1, that is } \frac{1}{|X|} \sum_{x \in X} f(x) = 1 \end{array} \right\}$$

Note that the center of Q is the all ones function: $\mathbb{1}(x) = 1$ for all $x \in X$.

Since we want to approximate the STSP with a projection of a polytope having not too many facets, this equates to approximating the set Q above with a section of a polytope having not too many vertices. We will view Q as living in the space L of all linear functions $f : X \rightarrow \mathbb{R}$, that is, restrictions of linear functions on \mathbb{R}^{n^2} to X . We in turn view L as living in \mathbb{R}^X , the space of all functions $f : X \rightarrow \mathbb{R}$.

With this setup, we have the following:

Theorem 1.1. *Let $n \geq 4$ be an integer and $k \leq n^{1/3}$ be an integer. There exists a polytope $P_k \subset \mathbb{R}^X$ with $O(n^{4k})$ vertices and a constant*

$$c_k = \frac{k}{n} + O\left(\frac{1}{n}\right)$$

such that

$$c_k(Q - \mathbb{1}) \subset P_k \cap L - \mathbb{1} \subset Q - \mathbb{1}$$

Note that the dimension of the convex hull of m vertices is $\leq m - 1$. Recall that we denoted by Z the center of the STSP. Thus, from the remarks above, this immediately gives as a corollary the following:

Theorem 1.2. *Let T_n be the Symmetric Traveling Salesman Polytope and let $k \leq n^{1/3}$ be an integer. Then there exists a polytope P_k° with $O(n^{4k})$ facets, a linear transformation T , and a constant*

$$c_k = \frac{k}{n} + O\left(\frac{1}{n}\right)$$

such that for $Q_k = T(P_k^\circ)$ we have

$$c_k(Q_k - Z) \subset T_n - Z \subset Q_k - Z$$

One notable aspect of this approximation is that the scaling factor gives us a *metric* bound on how far our approximating set can be from the STSP.

1.3 Computability Remarks

Say that P is a polytope living in \mathbb{R}^N with $O(N)$ facets and $\pi : \mathbb{R}^N \rightarrow \mathbb{R}^n$ is a projection. Then deciding if a point is in $\pi(P)$ becomes a linear programming problem in $O(N)$ equations and variables. Linear programming is decidable in time polynomial in the number of equations and variables (see, for example Chapter 3 of [3]).

2 Projection Construction

Recall that we consider Q to be a subset of \mathbb{R}^X , and we will be approximating Q by a polytope in \mathbb{R}^X with not too many vertices. Thus, we need to describe which functions in \mathbb{R}^X will serve as our vertices (the functions of which we will take the convex hull). Fixing a $k < n^{1/3}$ (the reason for this restriction will be evident later), we will consider functions, each indexed by a particular subset of the edges of the complete graph K_n . We only consider subsets of edges of K_n which could correspond to a subset of a Hamiltonian cycle in K_n ; namely, a subset of edges which correspond to disjoint paths. We call such subsets “path subsets.”

Given a path subset Γ with k edges in it, note that the lengths of the disjoint paths in Γ are a partition π of k . We call this partition the “partition type” of Γ . From the following Lemma, we can see that the number of Hamiltonian cycles containing all edges in Γ depends only on k and on the number of parts in π (i.e., on the partition type of Γ).

Lemma 2.1. *Let (k_1, k_2, \dots, k_m) be a partition of $k \leq n - 1$ ($k + m \leq n$) and K_n the complete graph on n vertices. Let p_1, p_2, \dots, p_m be disjoint paths in K_n of length k_1, \dots, k_m respectively. Then the number of Hamiltonian cycles in K_n containing all of paths p_1, \dots, p_m is:*

$$2^{m-1}(n - k - 1)!$$

Proof. Note that the restriction $k + m \leq n$ assures that it is possible to find disjoint paths in K_n of lengths k_1, \dots, k_m . Any cycle containing the paths p_1, \dots, p_m can be written uniquely as a sequence of numbers, beginning with path p_1 in a particular orientation. Thinking of the remaining paths as blocks with 2 orientations and the remaining numbers as blocks with a single orientation, we find that there are $2^{m-1}(n-k-1)!$ ways of ordering and orienting the remaining blocks. Each of these orders and orientations corresponds uniquely to a Hamiltonian cycle containing paths p_1, \dots, p_m . \square

Thus, we shall denote by $a_{\pi(\Gamma)} = a_\pi$ the number of Hamiltonian cycles containing all edges in Γ , where Γ has partition type π .

First we define $[A]$ to be the indicator function of a set A . That is,

$$[A](x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

By abuse of notation, for a single point x we write $[x]$ for its indicator function instead of $[\{x\}]$.

Now we can define

$$g_\Gamma = \frac{|X|}{a_{\pi(\Gamma)}} \sum_{\substack{x \in X \\ \Gamma \subset x}} [x]$$

Thus,

$$g_\Gamma(x) = \begin{cases} \frac{|X|}{a_{\pi(\Gamma)}} & \text{if } x \text{ contains all edges in } \Gamma \\ 0 & \text{otherwise} \end{cases}$$

Note that the constant is chosen so that g_Γ has an average of 1 on X .

Given a $k < n^{1/3}$ and a partition π of k , we will now describe a linear operator

$$T_\pi : \mathbb{R}^X \rightarrow \text{span}\{g_\Gamma : \Gamma \text{ is a path subset, } |\Gamma| = k, \Gamma \text{ has partition type } \pi\}$$

Define

$$\begin{aligned} A_\Gamma &= \{x \in X : x \text{ contains all the edges in } \Gamma\} \\ B_\pi &= \{\Gamma \text{ a path subset} : \Gamma \text{ has partition type } \pi\} \end{aligned}$$

Note that $|A_\Gamma| = a_{\pi(\Gamma)} = a_\pi$. For a $f \in \mathbb{R}^X$, we define

$$T_\pi(f) = \frac{1}{|B_\pi|} \sum_{\Gamma \in B_\pi} \alpha_\Gamma g_\Gamma$$

where

$$\alpha_\Gamma = \frac{1}{a_{\pi(\Gamma)}} \sum_{x \in A_\Gamma} f(x)$$

In words, the operator T_π is a weighted sum of the functions g_Γ where Γ has partition type π . The weight of a particular g_Γ is

$$\frac{\text{average value of } f \text{ on } x \text{ containing } \Gamma}{\text{number of } g_\Gamma \text{ with partition type } \pi}$$

Suppose that f is a function which has average value 1 on X . Then the sum of the above coefficients is:

$$\begin{aligned} \frac{1}{|B_\pi|} \sum_{\Gamma \in B_\pi} \frac{1}{a_\pi(\Gamma)} \sum_{x \in A_\pi(\Gamma)} f(x) &= \sum_{x \in X} \frac{|\{\text{path subsets in } x \text{ of partition type } \pi\}|}{|B_\pi| a_\pi} f(x) \\ &= \frac{1}{|X|} \sum_{x \in X} f(x) = 1 \end{aligned}$$

Thus, we can see that if f is a function which has average value 1 on X , then $T_\pi(f)$ is a convex combination of the g_Γ s. Our goal is to understand how T_π acts on the g_{st} defined as follows:

$$g_{st} = \frac{n-1}{2} \sum_{x \text{ contains edge } \{s,t\}} [x]$$

so that

$$g_{st}(x) = \begin{cases} \frac{(n-1)}{2} & \text{if } x \text{ contains edge } \{s, t\} \\ 0 & \text{otherwise} \end{cases}$$

Note that g_{st} is the particular case of g_Γ where Γ consists of a single edge.

It is clear that Q (defined in section 1.2) is contained in the affine span of the g_{st} . Our agenda at this point is to obtain a convex combination T of the linear maps T_π and a constant $c_k = \frac{k}{n} + O(\frac{1}{n})$ such that

$$T(g_{st}) = (1 - c_k)\mathbb{1} + (c_k)g_{st} \tag{2.1}$$

The linear map T will act the same way on the affine span of the g_{st} . We will define P_k to be the convex hull of the g_Γ used in T , and L to be the subspace of linear functions on X . Because each of the g_Γ are nonnegative on X and have average value 1 on X , we can see that any function in $P_k \cap L$ will also be in Q . And since, as mentioned already, Q is contained in the affine span of the g_{st} and T will act the same way on the affine span of the g_{st} , we will have:

$$c_k(Q - \mathbb{1}) \subset P_k \cap L - \mathbb{1} \subset Q - \mathbb{1}$$

which will give us Theorem 1.1. Thus, we proceed in finding such a linear map T .

Let π be a partition of $k < n^{1/3}$. Then

$$T_\pi(g_{st}) = \frac{1}{|B_\pi|} \sum_{\Gamma \in B_\pi} \left(\frac{1}{a_\pi(\Gamma)} \sum_{x \in A_\Gamma} g_{st}(x) \right) g_\Gamma \quad (2.2)$$

For a partition $\pi = (k_1, k_2, \dots, k_m)$ of k , we want to calculate exactly what the function $T_\pi(g_{st})$ does. In equation (2.2), we firstly note that g_{st} takes on only values 0 and $\frac{n-1}{2}$, and that g_Γ takes on only values 0 and $\frac{|X|}{a_\pi}$. Thus, we can see that

$$\begin{aligned} T_\pi(g_{st})(y) &= \frac{|X|(n-1)}{a_\pi 2 |B_\pi| a_\pi} \#\{\text{pairs } (x, \Gamma) \text{ such that } x \in A_\Gamma, g_{st}(x) = \frac{n-1}{2}, \\ &\quad g_\Gamma(y) = \frac{|X|}{a_\pi}\} \\ &= \frac{|X|(n-1)}{a_\pi 2 |B_\pi| a_\pi} \#\{x : \{s, t\} \in x, \Gamma \subset x \text{ for some } \Gamma \subset y \\ &\quad \text{of partition type } \pi\} \end{aligned} \quad (2.3)$$

Note that if Γ contains edges corresponding to a path of length ≥ 2 with s and t as its endpoints, there cannot be a Hamiltonian cycle containing all edges in Γ as well as the edge $\{s, t\}$ (recall that $k \leq n^{1/3}$). If Γ contains edges corresponding to a path where s is connected to two vertices, neither of which is t , there cannot be a Hamiltonian cycle containing all edges in Γ as well as the edge $\{s, t\}$ (and similarly for t) because any vertex in a Hamiltonian cycle has exactly 2 vertices adjacent to it. There are 4 ways in which a Hamiltonian cycle *can* contain all edges in Γ as well as the edge $\{s, t\}$:

1. $\{s, t\}$ is an edge in Γ .
2. s and t are each endpoints of different paths in Γ
3. Exactly one of s or t is an endpoint of a path in Γ , and the other does not appear in Γ
4. Neither s nor t appear in Γ

Suppose that Γ has partition type $\pi = (k_1, \dots, k_m)$. If $\{s, t\} \in \Gamma$, then $\Gamma \cup \{\{s, t\}\}$ again contains k edges and has a partition type with m parts. If s and t are each endpoints of separate paths in Γ , then $\Gamma \cup \{\{s, t\}\}$ contains $k+1$ edges and has a partition type with $m-1$ parts. If exactly one of s or t is an endpoint of a path in Γ , then $\Gamma \cup \{\{s, t\}\}$ contains $k+1$ edges and has a partition type with m parts. And if neither s nor t appear in Γ , then $\Gamma \cup \{\{s, t\}\}$ contains $k+1$ edges and has a partition type with $m+1$ parts.

Using Lemma 2.1 and equation (2.3), we can now more explicitly describe the function $T_\pi(g_{st})$.

$$\begin{aligned}
T_\pi(g_{st})(y) = & \frac{|X|(n-1)}{a_\pi 2^{|B_\pi|} a_\pi} \left(|\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } \{s, t\} \in \Gamma\}| \right. \\
& \cdot 2^{m-1}(n-k-1)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } s, t \text{ are each endpoints} \\
& \text{of different paths in } \Gamma\}| \cdot 2^{m-2}(n-k-2)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and exactly one of } s, t \text{ is an endpoint} \\
& \text{of a path in } \Gamma\}| \cdot 2^{m-1}(n-k-2)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi \text{ and } s, t \text{ are not in } \Gamma\}| \\
& \left. \cdot 2^m(n-k-2)! \right) \tag{2.4}
\end{aligned}$$

Thus, we can see that $T_\pi(g_{st})(y)$ depends only on how s and t sit in the paths of subsets $\Gamma \subset y$ of partition type π . In other words, the number of vertices between s and t in the Hamiltonian path y determines $T_\pi(g_{st})(y)$. The following lemma shows that $T_\pi(g_{st})$ takes on no more than $k+1$ values on X :

Lemma 2.2. *Let π be a partition of the positive integer $k \leq n^{1/3}$. Let m_y be the number of vertices between s and t in the shorter path between s and t in the Hamiltonian cycle y . Then for any y such that $m_y \geq k$, the value of $T_\pi(g_{st})(y)$ is the same.*

Thus, this Lemma implies that the range of values taken on by $T_\pi(g_{st})$ can be found by evaluating $T_\pi(g_{st})$ on Hamiltonian cycles having $0, 1, \dots, k$ vertices between s and t . The proof of this lemma is postponed until section 4.

We also know how $T_\pi(g_{st})$ acts on y which contain the edge (s, t) :

Lemma 2.3. *Let π be a partition of $k \leq n^{1/3}$ and suppose that y contains the edge $\{s, t\}$. Then $T_\pi(g_{st})(y) = \frac{k+2}{2} + O\left(\frac{1}{n^{2/3}}\right)$.*

Again we postpone the proof until section 4.

We note that Lemma 2.3 is key to the fact that it is feasible to find a convex combination of linear maps T_π , resulting in a map T acting as in equation (2.1). What we will show is that $T_\pi(g_{st})$ is “almost” $(1-a)\mathbb{1} + ag_{st}$ for $a \in (0, 1)$. It isn’t *exactly* $(1-a)\mathbb{1} + ag_{st}$ because $T_\pi(g_{st})$ isn’t the same value on all Hamiltonian cycles which do not contain the edge $\{s, t\}$. But the different values it takes on for Hamiltonian cycles not containing edge $\{s, t\}$ are very close to each other. Thus, we take a convex combination of T_π s for varying partitions π is to “smooth out” those differences, resulting in a single map T taking on only 2 values: a single value for Hamiltonian cycles containing $\{s, t\}$ and a different value for Hamiltonian cycles not containing $\{s, t\}$. This will imply that $T(g_{st}) = (1-a)\mathbb{1} + ag_{st}$ for some $a \in (0, 1)$ (recall, T_π maps functions with average value 1 to functions with average value 1). The T_π s used in the map T will involve partitions π of varying numbers. Using Lemma 2.3, we will find

that $T(g_{st})$ will be $\frac{k+1}{2}(1 + O(\frac{1}{n^{1/3}}))$ on Hamiltonian cycles containing the edge $\{s, t\}$. Thus,

$$(1 - a) + a\frac{n-1}{2} = \frac{k+1}{2}\left(1 + O\left(\frac{1}{n^{1/3}}\right)\right)$$

so that

$$a = \frac{k}{n} + O\left(\frac{1}{n}\right)$$

For each of the following Lemmas, we let y_i be a Hamiltonian cycle which has i vertices between vertex s and vertex t , and let y_{i+1} be a Hamiltonian cycle which has $i+1$ vertices between vertex s and vertex t . We let $n_{(m)}$ denote $n(n-1)(n-2)\dots(n-m+1)$.

Lemma 2.4. *Let $1 \leq i \leq k-1$. If π is the partition $\underbrace{(1, 1, \dots, 1)}_{k \text{ ones}}$ then*

$$T_\pi(g_{st})(y_{i+1}) - T_\pi(g_{st})(y_i) = (-1)^{i+1} \frac{(n-1)(n-2k)(n-2k-1)k_{(i+1)}}{4(n-k-1)n_{(i+2)}}$$

Lemma 2.5. *If π is the partition $(k-1, 1)$ for $k \geq 3$ then*

$$T_\pi(g_{st})(y_{i+1}) - T_\pi(g_{st})(y_i) = \begin{cases} -\frac{(n-1)(n-k-3)}{n(n-k-1)^2} & 1 \leq i \leq k-3 \\ \frac{3(n-1)}{2n(n-k-1)^2} & i = k-2 \\ \frac{(n-1)}{2n(n-k-1)^2} & i = k-1 \end{cases}$$

Lemma 2.6. *If π is the partition (k) then*

$$T_\pi(g_{st})(y_{i+1}) - T_\pi(g_{st})(y_i) = \begin{cases} -\frac{(n-1)}{n(n-k-1)} & 1 \leq i \leq k-2 \\ 0 & i = k-1 \end{cases}$$

The proofs of the above Lemmas are postponed to section 4. Using all of these Lemmas, we can prove Theorem 1.1.

Proof of Theorem 1.1. Recall from earlier comments that we need only find a convex combination T of the linear maps T_π such that

$$T(g_{st})(y) = \frac{k+1}{2}\left(1 + O\left(\frac{1}{n^{1/3}}\right)\right)$$

on Hamiltonian cycles y containing the edge $\{s, t\}$, and such that the number of functions g_Γ used in T is of order n^{4k} .

Recall that we assume $k < n^{1/3}$. The maps T_π which we will use will correspond to the partitions

$$\begin{aligned} \pi_\ell &= (\ell - 1, 1) & 1 \leq \ell \leq 2k \\ \pi_{\ell'} &= (\ell) & 2 \leq \ell \leq 2k + 1 \\ \pi^* &= \underbrace{(1, 1, \dots, 1)}_{2k \text{ } 1\text{s}} \end{aligned}$$

The way that T is obtained is as follows: We will use the T_{π_ℓ} and $T_{\pi_{\ell'}}$ to adjust T_{π^*} . Recall (from Lemma 2.4) that if y_{2k} is a Hamiltonian cycle with $2k$ vertices between s and t and y_{2k-1} is a Hamiltonian cycle with $2k-1$ vertices between s and t , then

$$\begin{aligned} & T_{\pi^*}(g_{st})(y_{2k}) - T_{\pi^*}(g_{st})(y_{2k-1}) \\ &= (-1)^{2k} \frac{(n-1)(n-2(2k))(n-2(2k)-1)(2k)_{(2k)}}{4(n-2k-1)n(n-2k-1)_{(2k+1)}} \end{aligned}$$

Since $2k$ is even, we define $a_{2k} = 0$ and find the positive b_{2k} such that

$$\begin{aligned} & T_{\pi^*}(g_{st})(y_{2k}) + a_{2k}T_{\pi_{2k}}(g_{st})(y_{2k}) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_{2k}) \\ & - (T_{\pi^*}(g_{st})(y_{2k-1}) + a_{2k}T_{\pi_{2k}}(g_{st})(y_{2k-1}) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_{2k-1})) = 0 \end{aligned}$$

From Lemmas 2.4 and 2.6, we can see that this would imply that $b_{2k} \sim \frac{(2k)_{(2k)}}{(n-2k-1)_{(2k-1)}}$, which is negligible if $k < n^{1/3}$. We can also see that, if y_{i+1} is a Hamiltonian cycle with $i+1$ vertices between s and t and y_i is a Hamiltonian cycle with i vertices between s and t for $i+1 < 2k$, then

$$\begin{aligned} & T_{\pi^*}(g_{st})(y_{i+1}) + a_{2k}T_{\pi_{2k}}(g_{st})(y_{i+1}) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_{i+1}) \\ & - (T_{\pi^*}(g_{st})(y_i) + a_{2k}T_{\pi_{2k}}(g_{st})(y_i) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_i)) \end{aligned}$$

is of the same order as

$$T_{\pi^*}(g_{st})(y_{i+1}) - T_{\pi^*}(g_{st})(y_i)$$

In the next step, since $2k-1$ is odd, we define $b_{2k-1} = 0$ and find the positive a_{2k-1} such that

$$\begin{aligned} & T_{\pi^*}(g_{st})(y_{2k-1}) + a_{2k-1}T_{\pi_{2k-1}}(g_{st})(y_{2k-1}) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_{2k-1}) \\ & - (T_{\pi^*}(g_{st})(y_{2k-2}) + a_{2k-1}T_{\pi_{2k-1}}(g_{st})(y_{2k-2}) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_{2k-2})) = 0 \end{aligned}$$

where y_{2k-1} is a Hamiltonian cycle with $2k-1$ vertices between s and t , and y_{2k-2} is a Hamiltonian cycle with $2k-2$ vertices between s and t . From Lemmas 2.4 and 2.5, we can see that this would imply that $a_{2k-1} \sim \frac{(2k-1)_{(2k-1)}}{(n-(2k-1)-1)_{(2k-3)}}$, which is negligible if $k > 3$ and $k < n^{1/3}$. We can also see that, if y_{i+1} is a Hamiltonian cycle with $i+1$ vertices between s and t and y_i is a Hamiltonian cycle with i vertices between s and t for $i+1 < 2k-1$, then

$$\begin{aligned} & T_{\pi^*}(g_{st})(y_{i+1}) + a_{2k-1}T_{\pi_{2k-1}}(g_{st})(y_{i+1}) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_{i+1}) \\ & - (T_{\pi^*}(g_{st})(y_i) + a_{2k-1}T_{\pi_{2k-1}}(g_{st})(y_i) + b_{2k}T_{\pi_{2k+1}'}(g_{st})(y_i)) \end{aligned}$$

is of the same order as

$$T_{\pi^*}(g_{st})(y_{i+1}) - T_{\pi^*}(g_{st})(y_i)$$

We continue this process, next smoothing out the values between Hamiltonian cycles having $2k - 2$ versus $2k - 3$ vertices between s and t . Since $T_{\pi^*}(g_{st})(y_{i+1}) - T_{\pi^*}(g_{st})(y_i)$ alternates sign, as we continue “smoothing out” $T_{\pi^*}(g_{st})$, we will alternately use T_{π_ℓ} and $T_{\pi_{\ell'}}$. All of the coefficients will be very small, except perhaps for the coefficient of T_{π_3} , which could be up to $\frac{1}{2}$ if k is close to $n^{1/3}$. Thus, when we divide by the sum of the coefficients (making a convex combination), the coefficient of T_{π^*} will be at least $\frac{1}{2} + O(\frac{1}{n^{1/3}})$. Thus, from Lemma 2.3, we know that the final T will have value $\frac{k+1}{2}(1 + O(\frac{1}{n^{1/3}}))$ on Hamiltonian cycles containing the edge $\{s, t\}$.

The number of different functions used in T will be the sum of the number of different functions used in T_{π^*} , T_{π_ℓ} , and $T_{\pi_{\ell'}}$. The number of different functions used in T_{π^*} is equal to the number of ways of picking $2k$ disjoint edges from the complete graph K_n ; i.e. the number of path subsets of partition type $(1, 1, \dots, 1)$. To pick $2k$ disjoint edges, we can pick $4k$ numbers from the set $\{1, 2, \dots, n\}$ in order. The first two we define as being an “edge”, the second two we define as being an “edge”, etc. Of course, we get the same set of edges if we picked two numbers which correspond to an edge in reverse order (i.e., instead of picking i , then $i + 1$ and defining them to be an edge, we picked $i + 1$ and then i and defined them to be an edge). We also get the same set of edges if two pairs of edges switch places in the ordering (i.e., instead of picking in order $i, i + 1, i + 2, i + 3$ and defining edges to be $\{i, i + 1\}$ and $\{i + 2, i + 3\}$, we had picked in order $i + 2, i + 3, i, i + 1$). Thus, we can see that the number of different functions used in T_π is

$$\frac{n^{(4k)}}{(2k)!2^{2k}} = O(n^{4k})$$

By similar arguments, we can see that the number of functions used in T_{π_ℓ} and $T_{\pi_{\ell'}}$ are of order smaller than n^{4k} . Thus, the total number of functions used in T is of order n^{4k} , and we have finished the proof. \square

3 Facets on the Boundary

Although there is no known complete description of the Symmetric Traveling Salesman Polytope as a system of linear inequalities, many facets are known (see, for example, chapter 58 of [8]). Some well-known facet defining inequalities are the following:

$$0 \leq x_{ij} \leq 1 \quad \text{for each } i, j \quad (3.1)$$

$$\sum_{\substack{j \in U \\ i \in V-U}} x_{ij} \geq 2 \quad \text{for each } U \subset V \text{ with } \emptyset \neq U \neq V \quad (3.2)$$

A natural question to ask regarding the approximation construction of the previous section would be: which (if any) of the above facets lie on the boundary

of the approximating set? In other words, which (if any) of the inequalities defining our approximating set coincide with one of the above inequalities?

Our construction creates a convex set P_k whose intersection with the space of linear functions lies inside of the dual of the STSP. Given our definition of the dual Q , we are looking for a function $f \in P_k$ which is linear and for which the set $\{x \in X : f(x) = 0\}$ is precisely the cycles for which equality holds in one of equations (3.1)-(3.2).

Fix some $k < n^{1/3}$. Recall that P_k is the convex hull of functions g_Γ which take a single positive value on cycles x containing the edges in Γ and 0 on cycles not containing the edges in Γ . For the functions used in P_k , Γ contains edges which correspond to a path of length $\ell \leq 2k + 1$, or a path of length $\ell \leq 2k - 1$ plus a single disjoint edge, or $2k$ disjoint edges.

Consider some edge $\{i, j\}$. Then the set $\Gamma = \{i, j\}$ corresponds to a path of length 1. We consider the function

$$f_{ij} = g_\Gamma$$

Note that $f_{ij} \in P_k$ for $k \geq 1$. Here we can see that $f_{ij}(x) = 0$ precisely when the cycle x does not contain the edge $\{i, j\}$. Also, it is clear that f_{ij} takes on the same value for each x containing the edge $\{i, j\}$. Thus, we can see that the f_{ij} s are each linear functions which correspond to the facets defined by the left hand sides of equations (3.1)

Again we consider some edge $\{i, j\}$. Now let

$$X_{ij} = \{\Gamma = \{\{i, a\}, \{i, b\}\} : a, b \in V; a \neq b; a, b \neq j\}$$

Note that X_{ij} consists of sets corresponding to paths of length 2. We define

$$f'_{ij} = \frac{1}{|X_{ij}|} \sum_{\Gamma \in X_{ij}} g_\Gamma$$

We can see that $f'_{ij} \in P_k$ for all $k \geq 1$. Note that f'_{ij} is nonzero on the Hamiltonian cycle x if and only if in x , i is adjacent to two vertices, neither of which is j ; i.e. if and only if i is not adjacent to j in x . Also note that if $f'_{ij}(x) \neq 0$, there is exactly one $\Gamma_x \in X_{ij}$ such that $g_{\Gamma_x}(x) \neq 0$. Thus, we can see that f'_{ij} is a constant multiple of the linear function $1 - x_{ij}$, which corresponds to a facet defined by the right hand side of equation (3.1).

Hence, we have shown that the facets defined by the left and right hand sides of equation (3.1) are on the boundary of P_k for $k \geq 1$.

Suppose we have some $U \subset V$ with $\emptyset \neq U \neq V$, $|U| \leq 2k$. For each $i < |U|$, let

$$X_i = \{\Gamma : \Gamma \text{ corresponds to a path of length } i + 1 \text{ with endpoints not in } U \\ \text{and } i \text{ vertices in } U\}$$

Recall that $g_\Gamma(x)$, $\Gamma \in X_i$ takes on two values; 0 if x does not contain Γ and a positive number depending only on the size and type of partition corresponding

to Γ if x does contain Γ . For each $i < |U|$ let c_i be a constant such that if $\Gamma_i \in X_i$ and x_i is a Hamiltonian cycle such that $g_{\Gamma_i}(x_i) \neq 0$, then

$$c_i g_{\Gamma_i}(x_i) = 2$$

and consider the function

$$h_U = \sum_{i=1}^{|U|-1} \frac{|U|-i}{|U|} c_i \sum_{\Gamma \in X_i} g_{\Gamma}$$

Let x be any Hamiltonian cycle. Note that there will be an even number of edges, say 2ℓ edges, from U to $V-U$ in x . These will correspond to ℓ paths with all vertices except the endpoints in U . If we sum over those ℓ paths the number of vertices that each of the paths has in U , we will get $|U|$. Those ℓ paths will correspond to the only $\Gamma \in X_i$ such that $g_{\Gamma}(x) \neq 0$. Thus, we can see that

$$h_U(x) = \sum_{\substack{j \in U \\ i \in V-U}} x_{ij} - 2$$

Hence, scaling h_U so that we have a convex combination of the g_{Γ} s, we can see that for $|U| \leq 2k$, the facets corresponding to (3.2) are on the boundary of P_k .

4 Proofs of Lemmas

Before we prove the Lemmas from section 2, we need one more Lemma.

Lemma 4.1. *Let m_1, \dots, m_p be nonnegative integers. Then the number of ways of picking m_1 paths of length 1, m_2 paths of length 2, \dots , m_p paths of length p all from a path of n vertices such that none of the chosen paths intersect is*

$$\frac{(n - \sum_{i=1}^p i m_i)_{(\sum_{i=1}^p m_i)}}{\prod_{i=1}^p m_i!}$$

Proof of Lemma 4.1. The value above gives the number of ways of coloring $\sum_{i=1}^p m_i$ of the numbers from 1 to $n - \sum_{i=1}^p i m_i$ so that m_i numbers are colored with color i . We will construct a unique set of paths as required by the Lemma from each such coloring, and show that any set of paths can be obtained by a coloring. We will present this bijection explicitly in the cases where we have exactly 1 or 2 total paths, and the cases where there are more paths will follow inductively.

Suppose we have a path P with n vertices in it; number the vertices $1, 2, \dots, n$ so that 1 and n are endpoint vertices, and i is adjacent to $i-1$ and $i+1$ for $1 < i < n$. For ease, we will think of P as if we can visualize it horizontally, so that vertex 1 is to the left of vertex 2, which is to the left of vertex 3, etc.

Fix a number $j, 1 \leq j \leq n-1$. Consider a set containing the numbers from 1 to $n-j$ such that one of these numbers is colored. To this coloring we

associate the path of length j in P which has its leftmost vertex located at the colored number. In other words, if i is the number which is colored in our set, we associate this coloring to the path of length j in P whose leftmost vertex is the vertex labeled i . Since there are j vertices to the right of the leftmost vertex in our path of length j , we see that this is a 1-1 correspondence between paths of length j within a path of n vertices and a coloring of one number from the set $\{1, 2, \dots, n - j\}$.

Now we consider a set containing the numbers from 1 to $n - j - i$ such that one of these numbers, say n_i , is colored with color i , and one, say n_j , is colored with color j . WLOG, suppose that $n_i < n_j$. If $n_j > n_i + i$, then to this coloring we associate the paths in P of length i and j such that the path of length i has the vertex labeled n_i as its leftmost vertex and the path of length j has the vertex labeled n_j as its leftmost vertex. A nice way to think of this is: given a coloring, we look at the path in P of length i with n_i as its leftmost vertex and the path in P of length j with n_j as its leftmost vertex. If these paths do not intersect (i.e. if $n_j > n_i + i$), then those are the paths we associate to the coloring. Otherwise, if they do intersect, we must move one of them.

Specifically, if $n_j = n_i + \ell$ where $\ell \leq i$, the paths intersect. In this case, we move the path of length j so that its leftmost vertex is $n - (j + \ell - 1)$. In other words, if there are ℓ' vertices between n_i and n_j , we move the path of length j in P so that if n'_j is the rightmost vertex of the moved path, there are exactly ℓ' vertices in P to the right of n'_j .

This association of colorings of two numbers between 1 and $n - j - i$ to placing paths of length i and j in P is clearly reversible. Specifically, suppose we have a path of length i in P with leftmost vertex n_i and a path of length j in P with leftmost vertex n_j which do not intersect. If n_i and n_j are both less than $n - i - j$, the associated coloring is: color n_i color i and color n_j color j . Otherwise, one of them is larger than $n - i - j$ (note that not both of them can be larger, because the two paths do not intersect). WLOG, say $n_j > n - i - j$. Then there are $\ell < i$ vertices to the right of the path of length j . So we color vertex n_i with color i , and vertex $n_i + \ell + 1$ with color j .

The association between colorings and path placings when we have more than 2 paths to place is a simple iteration of the procedure when there are 1 or 2 paths to place. Specifically, suppose we have a coloring of the numbers between 1 and $n - \sum_{i=1}^p im_i$ so that m_i numbers are colored with color i . From this coloring, we get a path placement as follows: let n_i be the smallest colored number, colored with color i . Place a path of length i in P with its leftmost vertex at vertex n_i . Remove n_i from the list of colored numbers. Label n as the "last" vertex in P . Now at each step remaining, we do the following: find n_j the smallest of the remaining colored numbers. If vertex n_j does not intersect one of the previously placed paths, place a path of length j with leftmost vertex at n_j . Then remove n_j from the list of colored numbers and label $n - i$ as the "last" vertex (where i is such that the closest colored vertex to the left of n_j is colored with color i). Otherwise, say n_ℓ is the leftmost vertex of a previously placed path of length ℓ and $n_j \leq n_\ell + \ell$. Let n_{last} be the vertex labeled "last" and $n'_j = n_{last} - j + (n_\ell - n_j) - 1$. Then we place a path of length j with leftmost

vertex n'_j . Remove n_j from the list of colored numbers, and label $n'_j - 1$ as the “last” vertex. Continue inductively. From the way that the “last” vertex is changed and from the fact that this is a 1-1 correspondence when we have 1 or 2 paths, we can see that this general construction works inductively. Hence, we have proven the Lemma. □

Proof of Lemma 2.3. Let π be a partition of k with m_i parts of size i , so that $\sum_i im_i = k$. Let $m = \sum_i m_i$ be the number of parts in π . Let y be a hamiltonian cycle which contains the edge $\{s, t\}$. Recall equation (2.4)

$$\begin{aligned}
T_\pi(g_{st})(y) = & \frac{|X|(n-1)}{a_\pi 2^m |B_\pi| a_\pi} \left(|\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } \{s, t\} \in \Gamma\}| \right. \\
& \cdot 2^{m-1}(n-k-1)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } s, t \text{ are each endpoints} \\
& \text{of different paths in } \Gamma\}| \cdot 2^{m-2}(n-k-2)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and exactly one of } s, t \text{ is an endpoint} \\
& \text{of a path in } \Gamma\}| \cdot 2^{m-1}(n-k-2)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi \text{ and } s, t \text{ are not in } \Gamma\}| \\
& \left. \cdot 2^m(n-k-2)! \right)
\end{aligned}$$

By the same argument as in the proof of Theorem 1.1, we can see that in this case

$$|B_\pi| = \frac{n(\sum_i (i+1)m_i)}{2^m \prod_i m_i!} = \frac{n(k+m)}{2^m \prod_i m_i!}$$

Also, using Lemma 2.1, we can calculate

$$a_\pi = 2^{m-1}(n-k-1)!$$

Thus, we can see that

$$\begin{aligned}
T_\pi(g_{st})(y) &= \frac{(n-1)_{(k)}(n-1) \prod_i m_i!}{2n_{(k+m)}} \frac{1}{2^{m-1}(n-k-1)!} \left(|\{\Gamma \subset y : \Gamma \text{ has} \right. \\
&\quad \text{partition } \pi, \text{ and } \{s, t\} \in \Gamma\}| \cdot 2^{m-1}(n-k-1)! \\
&\quad + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } s, t \text{ are each endpoints} \\
&\quad \text{of different paths in } \Gamma\}| \cdot 2^{m-2}(n-k-2)! \\
&\quad + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and exactly one of } s, t \text{ is an endpoint} \\
&\quad \text{of a path in } \Gamma\}| \cdot 2^{m-1}(n-k-2)! \\
&\quad \left. + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi \text{ and } s, t \text{ are not in } \Gamma\}| \right. \\
&\quad \left. \cdot 2^m(n-k-2)! \right) \\
&\geq \frac{(n-1)_{(k)}(n-1) \prod_i m_i!}{2n_{(k+m)}} \frac{1}{2^{m-1}(n-k-1)!} \left(|\{\Gamma \subset y : \Gamma \text{ has} \right. \\
&\quad \text{partition } \pi, \text{ and } \{s, t\} \in \Gamma\}| \cdot 2^{m-1}(n-k-1)! \\
&\quad \left. + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi \text{ and } s, t \text{ are not in } \Gamma\}| \right. \\
&\quad \left. \cdot 2^m(n-k-2)! \right)
\end{aligned}$$

We can count $|\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } \{s, t\} \in \Gamma\}|$ as follows: suppose that $m_i \neq 0$ (this is true for at least some i). Mark a path of length i arbitrarily in the Hamiltonian cycle y . The number of ways that the remaining paths can be chosen, using Lemma 4.1, is

$$\frac{(n - (i + 1) - \sum_{j \neq i} j m_j - (m_i - 1)i)_{((m_i - 1) + \sum_{j \neq i} m_j)}}{(m_i - 1)! \prod_{j \neq i} m_j!} = \frac{(n - k - 1)_{(m-1)}}{(m_i - 1)! \prod_{j \neq i} m_j!}$$

Then there are k ways of rotating the cycle y cyclically so that the edge $\{s, t\}$ lies in one of our chosen paths. Since the m_i paths of length i are indistinguishable, we need to divide by m_i in order to not overcount. Thus, we have found

$$|\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } \{s, t\} \in \Gamma\}| = \frac{k(n - k - 1)_{(m-1)}}{\prod_j m_j!}$$

Using Lemma 4.1, we can calculate

$$|\{\Gamma \subset y : \Gamma \text{ has partition } \pi \text{ and } s, t \text{ are not in } \Gamma\}| = \frac{(n - 2 - k)_{(m)}}{\prod_j m_j!}$$

Thus, since we assume $k < n^{1/3}$, we have shown that

$$\begin{aligned}
T_\pi(g_{st})(y) &\geq \frac{(n-1)_{(k)}(n-1) \prod_i m_i!}{2n_{(k+m)}} \frac{1}{2^{m-1}(n-k-1)!} \\
&\quad \cdot \left(\frac{k(n-k-1)_{(m-1)}}{\prod_j m_j!} \cdot 2^{m-1}(n-k-1)! \right. \\
&\quad \left. + \frac{(n-2-k)_{(m)}}{\prod_j m_j!} \cdot 2^m(n-k-2)! \right) \\
&= \frac{k+2}{2} + O\left(\frac{1}{n^{2/3}}\right)
\end{aligned}$$

□

Proof of Lemma 2.2. Looking at equation (2.4), we see that we can think of $T_\pi(g_{st})(y)$ as a sum over all $\Gamma \subset y$ with partition type π , each Γ contributing a certain amount (maybe 0). We need to show that $T_\pi(g_{st})(y)$ is the same for all Hamiltonian cycles y such that the distance between s and t in y is at least k . We will do this by showing that if y is a Hamiltonian cycle with d vertices between s and t , $k \leq d < \frac{n}{2}$, then for a Hamiltonian cycle y' with $d+1$ vertices between s and t we have $T_\pi(g_{st})(y) = T_\pi(g_{st})(y')$. Since the only thing that affects the value of $T_\pi(g_{st})(y)$ is the number of vertices between s and t in y , we can WLOG consider the following two cases

$$\begin{array}{ll}
y = (1, 2, \dots, n) & s = 1, \quad t = d \\
y' = (1, 2, \dots, n) & s' = 1, \quad t' = d+1
\end{array}$$

where $d \geq k+2$. (Hence we will show $T_\pi(g_{st})(y) = T_\pi(g_{s't'})(y')$.)

Consider any $\Gamma \subset y$ with partition type π . Consider the exact same $\Gamma \subset y'$. Note that, since π is a partition of k and there are at least $k+1$ edges from vertex 1 to vertex d (at least $k+2$ edges from vertex 1 to vertex $d+1$), Γ cannot have a path in it connecting vertex 1 to vertex d (connecting vertex 1 to vertex $d+1$). Suppose that one of the following is true:

1. d and $d+1$ are each endpoints of some path in Γ
2. Neither d nor $d+1$ appears in Γ
3. Both d and $d+1$ are in the middle (*not* an endpoint) of a path in Γ

Then, in looking at equation (2.4), we see that this particular Γ contributes the same amount in $T_\pi(g_{st})(y)$ as in $T_\pi(g_{s't'})(y')$. The only cases where Γ contributes differing amounts in $T_\pi(g_{st})(y)$ and $T_\pi(g_{s't'})(y')$ are:

1. One of d or $d+1$ is an endpoint of a path in Γ , and the other does not appear in Γ
2. One of d or $d+1$ is an endpoint of a path in Γ , and the other is in the middle (*not* an endpoint) of a path in Γ .

If Γ lies in one of those two cases, it contributes a different amount in $T_\pi(g_{st})(y)$ versus in $T_\pi(g_{s't'})(y')$. Thus, to each Γ in one of those two cases, we will associate a unique Γ' (also in one of those two cases) for which the contribution of Γ in $T_\pi(g_{st})(y)$ is equal to the contribution of Γ' in $T_\pi(g_{s't'})(y')$, and the contribution of Γ' in $T_\pi(g_{st})(y)$ is equal to the contribution of Γ in $T_\pi(g_{s't'})(y')$. We will show that if the partner to Γ under this association is Γ' , then the partner to Γ' under this association is Γ . Once we have this, we will be done.

Given some Γ in one of the two cases above, consider vertices $d-1$ and $d+2$. If vertex $d-1$ is not connected to vertex $d-2$ and vertex $d+2$ is not connected to vertex $d+3$, we map $\Gamma \mapsto \Gamma'$, where Γ' leaves all paths in Γ untouched, except for the paths from $d-1$ to $d+2$ which it “reflects” about the line between d and $d+1$, as demonstrated in the following picture:

$$\begin{array}{ccc} \circ_{d-1} - \circ_d & \circ_{d+1} & \circ_{d+2} & \leftrightarrow & \circ_{d-1} & \circ_d & \circ_{d+1} - \circ_{d+2} \\ & & & & & & \\ & & & & \text{and} & & \\ \circ_{d-1} - \circ_d - \circ_{d+1} & \circ_{d+2} & & \leftrightarrow & \circ_{d-1} & \circ_d - \circ_{d+1} - \circ_{d+2} \end{array}$$

Since vertex $d-1$ is not connected to vertex $d-2$ and vertex $d+2$ is not connected to vertex $d+3$, this action preserves the partition type so that we obtain a partner Γ' again of partition type π . It is also clear that this action indeed produces a Γ' such that the contribution of Γ in $T_\pi(g_{st})(y)$ is equal to the contribution of Γ' in $T_\pi(g_{s't'})(y')$, and the contribution of Γ' in $T_\pi(g_{st})(y)$ is equal to the contribution of Γ in $T_\pi(g_{s't'})(y')$. We also note that Γ' is mapped to Γ under this action.

If $d-1$ was connected to $d-2$ or if $d+2$ was connected to $d+3$, then check to see if $d-2$ is connected to $d-3$ and if $d+3$ is connected to $d+4$. If $d-2$ is not connected to $d-3$ and $d+3$ is not connected to $d+4$, we can do the same “reflecting” action, this time between the paths from $d-2$ to $d+3$. If not, continue looking for the first place where we can reflect.

Firstly we note that the first place to reflect is well-defined, and that if this action associates Γ to Γ' , it associates Γ' to Γ . Secondly, we note that we will find a “first place to reflect” before getting down to vertex 1. This is because $d \geq k+2$ and there are only k edges in Γ . Finally, it is clear that this action indeed produces a Γ' such that the contribution of Γ in $T_\pi(g_{st})(y)$ is equal to the contribution of Γ' in $T_\pi(g_{s't'})(y')$, and the contribution of Γ' in $T_\pi(g_{st})(y)$ is equal to the contribution of Γ in $T_\pi(g_{s't'})(y')$. Thus, we have proven the Lemma. □

The ideas of “reflecting” in the proof of this lemma will be also come into play in the proofs of Lemmas 2.4-2.6.

Proof of Lemma 2.4. The proof of Lemma 2.2 shows us exactly how to prove Lemma 2.4. Namely, the reason that $T_\pi(g_{st})$ takes on different values for Hamiltonian cycles having i and $i+1$ vertices between s and t ($i < k$) is because the

association from Γ to Γ' may not work. In particular, let π be the partition $\underbrace{(1, 1, \dots, 1)}_{k \text{ 1s}}$ and

$$\begin{aligned} y &= (1, 2, \dots, n) & s &= 1, \quad t = i + 2 \\ y' &= (1, 2, \dots, n) & s' &= 1, \quad t' = i + 3 \end{aligned}$$

where $1 \leq i < k - 1$. (Then y has i vertices between s and t , y' has $i + 1$ vertices between s and t). Then each path subset Γ of partition type π can be associated to a Γ' just as in the proof of Lemma 2.2, unless we have something like the following:

$$\circ_1 - \circ_2 \quad \circ_3 - \circ_4 \quad \dots \quad \circ_{i+1} - \circ_{i+2} \quad \circ_{i+3} \quad \circ_{i+4} - \circ_{i+5} \dots \circ_{2i+2} - \circ_{2i+3} \quad (4.1)$$

or

$$\circ_1 \quad \circ_2 - \circ_3 \quad \circ_4 - \circ_5 \dots \circ_i - \circ_{i+1} \quad \circ_{i+2} \quad \circ_{i+3} - \circ_{i+4} \dots \circ_{2i+3} - \circ_{2i+4} \quad (4.2)$$

in the case of i even or

$$\circ_1 - \circ_2 \quad \circ_3 - \circ_4 \quad \dots \quad \circ_i - \circ_{i+1} \quad \circ_{i+2} \quad \circ_{i+3} - \circ_{i+4} \dots \circ_{2i+2} - \circ_{2i+3} \quad (4.3)$$

or

$$\circ_1 \quad \circ_2 - \circ_3 \quad \circ_4 - \circ_5 \dots \circ_{i+1} - \circ_{i+2} \quad \circ_{i+3} \quad \circ_{i+4} - \circ_{i+5} \dots \circ_{2i+3} - \circ_{2i+4} \quad (4.4)$$

in the case of i odd.

Thus, we can see that the *difference* between $T_\pi(g_{st})(y)$ and $T_\pi(g_{s't'})(y')$ is simply the difference in the contribution of each path subset Γ which does not have a valid partner to which it can map (i.e., if Γ corresponds to one of the above cases). Thus, in looking at equation (2.4) and using the notation of Lemma 2.4, we can see that $T_\pi(g_{st})(y) - T_\pi(g_{s't'})(y')$ is

$$\begin{aligned} & \frac{|X|(n-1)}{a_\pi 2^{|B_\pi|} a_\pi} \left(|\{\Gamma \text{ in case of (4.2) and } \Gamma \text{ contains edge } \{1, n\}\}| \right. \\ & \quad \cdot 2^{k-2}(n-k-2)! \\ & \quad + |\{\Gamma \text{ in case of (4.2) and } \Gamma \text{ does not contain edge } \{1, n\} \\ & \quad \quad \text{or } \Gamma \text{ in case of (4.1)}\}| \cdot 2^{k-1}(n-k-2)! \\ & \quad - |\{\Gamma \text{ in case of (4.1)}\}| \cdot 2^{k-2}(n-k-2)! \\ & \quad - |\{\Gamma \text{ in case of (4.2) and } \Gamma \text{ contains edge } \{1, n\}\}| \cdot 2^{k-1}(n-k-2)! \\ & \quad - |\{\Gamma \text{ in case of (4.2) and } \Gamma \text{ does not contain edge } \{1, n\}\}| \\ & \quad \left. \cdot 2^k(n-k-2)! \right) \end{aligned}$$

when i is even and

$$\begin{aligned}
& \frac{|X|(n-1)}{a_\pi 2|B_\pi|a_\pi} \left(|\{\Gamma \text{ in case of (4.3)}\}| \cdot 2^{k-2}(n-k-2)! \right. \\
& \quad + |\{\Gamma \text{ in case of (4.4) and } \Gamma \text{ contains edge } \{1, n\}\}| \cdot 2^{k-1}(n-k-2)! \\
& \quad + |\{\Gamma \text{ in case of (4.4) and } \Gamma \text{ does not contain edge } \{1, n\}\}| \\
& \quad \quad \cdot 2^k(n-k-2)! \\
& \quad - |\{\Gamma \text{ in case of (4.4) and } \Gamma \text{ contains edge } \{1, n\}\}| \cdot 2^{k-2}(n-k-2)! \\
& \quad - |\{\Gamma \text{ in case of (4.4) and } \Gamma \text{ does not contain edge } \{1, n\} \\
& \quad \quad \text{or } \Gamma \text{ in case of (4.3)}\}| \cdot 2^{k-1}(n-k-2)! \\
& \left. \right)
\end{aligned}$$

when i is odd.

Using Lemma 4.1, we can actually calculate these differences. In the case of i even, we have

$$\begin{aligned}
& \frac{|X|(n-1)}{a_\pi 2|B_\pi|a_\pi} \left(\frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}}{(k-i-2)!} 2^{k-2}(n-k-2)! \right. \\
& \quad + \left(\frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} \right. \\
& \quad \left. + \frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} \right) \cdot 2^{k-1}(n-k-2)! \\
& \quad - \frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} 2^{k-2}(n-k-2)! \\
& \quad - \frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}}{(k-i-2)!} 2^{k-1}(n-k-2)! \\
& \quad \left. - \frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} 2^k(n-k-2)! \right) \quad (4.5)
\end{aligned}$$

and in the case of i odd we have

$$\begin{aligned}
& \frac{|X|(n-1)}{a_\pi 2|B_\pi|a_\pi} \left(\frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} 2^{k-2}(n-k-2)! \right. \\
& + \frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}}{(k-i-2)!} 2^{k-1}(n-k-2)! \\
& + \frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} 2^k(n-k-2)! \Big) \\
& - \frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}}{(k-i-2)!} 2^{k-2}(n-k-2)! \\
& - \left(\frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} \right. \\
& \left. - \frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} \right) \cdot 2^{k-1}(n-k-2)! \quad (4.6)
\end{aligned}$$

Thus, to calculate these values, all we have left is to compute the values of $|X|$, a_π , and $|B_\pi|$. We have already identified that $|X|$, the number of Hamiltonian cycles in the complete graph on n vertices K_n , is $\frac{(n-1)!}{2}$. Recall that $|B_\pi|$ is the number of path subsets of partition type π . By the same argument used in the proof of Theorem 1.1, we have:

$$|B_\pi| = \frac{n(2k)}{2^k k!}$$

Finally, recall that a_π is the number of Hamiltonian cycles containing all edges in a path subset Γ of partition type π . Then from Lemma 2.1, we know that for $\pi = \underbrace{(1, 1, \dots, 1)}_{k \text{ 1s}}$, $a_\pi = 2^{k-1}(n-k-1)!$. Plugging all of these into equations (4.5) and (4.6), we find that for i even we have

$$\begin{aligned}
& \frac{(n-1)_{(k)}(n-1)k!}{n_{(2k)}(n-k-1)4} \left(\frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}}{(k-i-2)!} \right. \\
& \quad + \frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}_2}{(k-i-1)!} \\
& \quad + \frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}_2}{(k-i-1)!} \\
& \quad - \frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} \\
& \quad - \frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}_2}{(k-i-2)!} \\
& \quad \left. - \frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}_4}{(k-i-1)!} \right) \\
& = \frac{(n-1)_{(k)}(n-1)k!}{n_{(2k)}(n-k-1)4} \left(-2 \frac{(n-k-i-3)_{(k-i-1)}}{(k-i-1)!} \right. \\
& \quad + \frac{(n-k-i-2)_{(k-i-1)}}{(k-i-1)!} \\
& \quad \left. - \frac{(n-k-i-3)_{(k-i-2)}}{(k-i-2)!} \right) \\
& = -\frac{(n-1)(n-2k)(n-2k-1)k_{(i+1)}}{4(n-k-1)n(n-k-1)_{(i+2)}} \tag{4.7}
\end{aligned}$$

and for i odd we have

$$\begin{aligned}
& \frac{(n-1)_{(k)}(n-1)k!}{n_{(2k)}(n-k-1)4} \left(\frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}}{(k-i-1)!} \right. \\
& \quad + \frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}_2}{(k-i-2)!} \\
& \quad + \frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}_4}{(k-i-1)!} \\
& \quad - \frac{(n-(2i+5)-(k-i-2))_{(k-i-2)}}{(k-i-2)!} \\
& \quad - \frac{(n-(2i+4)-(k-i-1))_{(k-i-1)}_2}{(k-i-1)!} \\
& \quad \left. - \frac{(n-(2i+3)-(k-i-1))_{(k-i-1)}_2}{(k-i-1)!} \right) \\
& = \frac{(n-1)_{(k)}(n-1)k!}{n_{(2k)}(n-k-1)4} \left(2 \frac{(n-k-i-3)_{(k-i-1)}}{(k-i-1)!} \right. \\
& \quad - \frac{(n-k-i-2)_{(k-i-1)}}{(k-i-1)!} \\
& \quad \left. + \frac{(n-k-i-3)_{(k-i-2)}}{(k-i-2)!} \right) \\
& = \frac{(n-1)(n-2k)(n-2k-1)k_{(i+1)}}{4(n-k-1)n(n-k-1)_{(i+2)}} \tag{4.8}
\end{aligned}$$

and we have finished our proof. \square

Proof of Lemma 2.5. Recall that here $\pi = (k-1, 1)$ and $k \geq 3$. In this case, we can actually calculate the value of $T_\pi(g_{st})$ on any Hamiltonian cycle. Recall that

$$\begin{aligned}
T_\pi(g_{st})(y) &= \frac{|X|(n-1)}{a_\pi 2 |B_\pi| a_\pi} \left(|\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } \{s, t\} \in \Gamma\}| \right. \\
& \quad \cdot 2^{m-1}(n-k-1)! \\
& \quad + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } s, t \text{ are each endpoints} \\
& \quad \text{of different paths in } \Gamma\}| \cdot 2^{m-2}(n-k-2)! \\
& \quad + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and exactly one of } s, t \text{ is an endpoint} \\
& \quad \text{of a path in } \Gamma\}| \cdot 2^{m-1}(n-k-2)! \\
& \quad \left. + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi \text{ and } s, t \text{ are not in } \Gamma\}| \right. \\
& \quad \left. \cdot 2^m(n-k-2)! \right) \tag{4.9}
\end{aligned}$$

Using the same argument used in the proof of Theorem 1.1, we can calculate

$$|B_\pi| = \frac{n^{(k+2)}}{4}$$

From Lemma 2.1 we can calculate

$$a_\pi = 2(n - k - 1)!$$

Suppose that y_i has i vertices between s and t for $1 \leq i \leq k - 2$. Then, using Lemma 4.1 and counting all the ways that the different intersection patterns described in (4.9) can occur, we calculate

$$\begin{aligned} T_\pi(g_{st})(y_i) &= \frac{4(n-1)(n-1)!}{16n_{(k+2)}(n-k-1)!(n-k-1)!} (4(n-k-2)! + (2(n-k-3) \\ &\quad + 2(n-k-i-2) + 2(n-k-i-1))2(n-k-2)! \\ &\quad + ((i-1)(n-k-i-1) + (n-k-i-2)(n-k-i-3)) \\ &\quad \cdot 4(n-k-2)!) \\ &= \frac{n-1}{2n(n-k-1)} \frac{2n^2 - (4k+2i+6)n + 2k^2 + 2ik + 6i + 6k + 4}{n-k-1} \end{aligned}$$

Suppose y_{k-1} has $k-1$ vertices between s and t . Then we calculate

$$\begin{aligned} T_\pi(g_{st})(y_{k-1}) &= \frac{4(n-1)(n-1)!}{16n_{(k+2)}(n-k-1)!(n-k-1)!} (6(n-k-2)! + (2(n-k-3) \\ &\quad + 2(n-k-2) + 2(n-2k) + 2(n-2k-1))2(n-k-2)! \\ &\quad + ((k-2)(n-2k) + (n-2k-1)(n-2k-2))4(n-k-2)!) \\ &= \frac{n-1}{2n(n-k-1)} \frac{2n^2 - (6k+2)n + 4k^2 + 8k - 5}{n-k-1} \end{aligned}$$

Finally, suppose that y_k has k vertices between s and t . Then we calculate

$$\begin{aligned} T_\pi(g_{st})(y_k) &= \frac{4(n-1)(n-1)!}{16n_{(k+2)}(n-k-1)!(n-k-1)!} (8(n-k-2)! + (4(n-k-3) \\ &\quad + 4(n-2k-1))2(n-k-2)! + ((k-1)(n-2k-1) \\ &\quad + (n-2k-2)(n-2k-3) + (n-k-3))4(n-k-2)!) \\ &= \frac{n-1}{2n(n-k-1)} \frac{2n^2 - (6k+2)n + 4k^2 + 8k - 4}{n-k-1} \end{aligned}$$

The proof now follows. \square

Proof of Lemma 2.6. Recall that here $\pi = (k)$. Again we can calculate the value of $T_\pi(g_{st})$ on any Hamiltonian cycle. Suppose that y_i has i vertices between s

and t for $1 \leq i \leq k-1$. Recall that

$$\begin{aligned}
T_\pi(g_{st})(y) = & \frac{|X|(n-1)}{a_\pi 2^{|B_\pi|} a_\pi} \left(|\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } \{s, t\} \in \Gamma\}| \right. \\
& \cdot 2^{m-1}(n-k-1)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and } s, t \text{ are each endpoints} \\
& \text{of different paths in } \Gamma\}| \cdot 2^{m-2}(n-k-2)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi, \text{ and exactly one of } s, t \text{ is an endpoint} \\
& \text{of a path in } \Gamma\}| \cdot 2^{m-1}(n-k-2)! \\
& + |\{\Gamma \subset y : \Gamma \text{ has partition } \pi \text{ and } s, t \text{ are not in } \Gamma\}| \\
& \left. \cdot 2^m(n-k-2)! \right) \tag{4.10}
\end{aligned}$$

Using the same argument used in the proof of Theorem 1.1, we can calculate

$$|B_\pi| = \frac{n(k+1)}{2}$$

From Lemma 2.1 we can calculate

$$a_\pi = (n-k-1)!$$

Thus, using Lemma 4.1 and counting all the ways that the different intersection patterns described in (4.10) can occur, we calculate

$$\begin{aligned}
T_\pi(g_{st})(y_i) &= \frac{2(n-1)(n-1)!}{4n_{(k+1)}(n-k-1)!(n-k-1)!} (2(n-k-2)! \\
& \quad + (n-k-i-2)2(n-k-2)!) \\
&= \frac{(n-1)(n-k-i-1)}{n(n-k-1)}
\end{aligned}$$

Suppose y_k has k vertices between s and t . Then we calculate

$$\begin{aligned}
T_\pi(g_{st})(y_k) &= \frac{2(n-1)(n-1)!}{4n_{(k+1)}(n-k-1)!(n-k-1)!} (4(n-k-2)! \\
& \quad + (n-2k-2)2(n-k-2)!) \\
&= \frac{(n-1)(n-2k)}{n(n-k-1)}
\end{aligned}$$

The proof now follows. □

5 Remaining Comments

Let π^* , π_ℓ , and $\pi_{\ell'}$ be as in the proof of Theorem 1.1. We note that we proved Lemmas 2.5 and 2.6 by calculating the values of $T_{\pi_\ell}(g_{st})$ and $T_{\pi_{\ell'}}(g_{st})$. One

can use Lemma 4.1 to calculate the value of $T_{\pi^*}(g_{st})$ on Hamiltonian cycles having 1 vertex between s and t , and use Lemma 2.4 to calculate the remaining values of $T_{\pi^*}(g_{st})$. Calculating values of $T_{\phi}(g_{st})$ for arbitrary partitions ϕ can be done using equation (2.4) and Lemma 4.1. However, determining the leading terms of (2.4) for arbitrary partitions and arbitrary Hamiltonian cycles is much more complicated when k is not fixed. It is a reasonable question whether one can find a convex combination of T_{ϕ} for other partitions ϕ which gives a better approximation, or which is valid for k closer to n .

The role of the linear operators T_{π} in the proof of Theorem 1.1 is to show that a scaling of the set $Q - \mathbb{1}$ lies inside of the set $P_k \cap L - \mathbb{1}$. It is entirely possible that the scaling factor we achieve using these linear operators is not optimal; that we could scale $Q - \mathbb{1}$ by a larger number and have it still lie inside $P_k \cap L - \mathbb{1}$.

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