Inequalities Concerning Eigenvalues
In all that follows, we will denote by $\theta$, the zero column vector and the identity matrix by $I_n$. Let $A = (a_{i,j})$ be an $n \times n$ real or complex matrix; the set of eigenvalues of $A$

$$\sigma(A) = \{ \lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n \}, \text{ where } |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| \geq |\lambda_n|$$

is called the spectrum of $A$. The eigenvalue $|\lambda_1|$ with the largest modulus is called the maximal (or dominant) eigenvalue. The spectral radius of $A$ denoted by $\rho(A)$ is the modulus of a maximal eigenvalue $|\lambda_1|$.

A matrix norm is defined as:

$$|| A || = \{ \max || A v ||; || v || = 1 \}.$$ 

For $i = 1, 2, \ldots, n$, define:

$$R_i(A) = \sum_{j=1}^{n} |a_{i,j}|; \quad r_i(A) = R_i(A) - |a_{i,i}| \quad \text{and} \quad C_j(A) = \sum_{i=1}^{n} |a_{i,j}|; \quad c_j(A) = C_j(A) - |a_{j,j}|.$$ 

The row norm or the infinity norm of $A$ is defined as follows:

$$|| A ||_\infty = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |a_{i,j}| \right\} = \max_{1 \leq i \leq n} R_i(A).$$ 

The column norm or the 1-norm of $A$ is defined as follows:

$$|| A ||_1 = \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} a_{i,j} \right\} = \max_{1 \leq j \leq n} C_j(A).$$ 

An orthogonal matrix $Q$ is a matrix satisfying $Q^t Q = I_n$ and a unitary matrix $U$ is a matrix with $U^* U = I_n$.

Lemma 1. \[ \det(A) = \prod_{i=1}^{n} \lambda_i \quad \text{and} \quad \text{trace}(A) = \sum_{i=1}^{n} \lambda_i. \]

Note that similar matrices have the same determinant and trace. Schur’s lemma. Any $n \times n$ matrix $A$ is unitarily similar to an upper triangular matrix $T = (t_{ij})$, i.e., $T = U^* A U$ for some unitary matrix $U$. 
Now we can present some important inequalities concerning eigenvalues.

**Theorem 1.** For any multiplicative matrix norm $\| \cdot \|$, we have $\rho(A) < \| A \|$.

**Proof.** Suppose $Au = \lambda u$, where $u$ is a unit vector (i.e., $\|u\| = 1$). Then

$$|\lambda| = |\lambda| \|u\| = \|Au\| \leq \max \{ \| Av \| : \| v \| = 1 \} = \| A \|.$$ 

Thus $\rho(A) \leq \| A \|$.

**Levy-Desplanques Theorem.** If the matrix $A = (a_{i,j})$ is strictly diagonally dominant, that is

$$|a_{i,i}| > r_i(A) \quad \forall i = 1, 2, \ldots, n.$$ 

Then $A$ is invertible.

**Proof.** Suppose $\det(A) = 0$, then for some nonzero vector $u = (u_1, u_2, \ldots, u_n)^t$, $Au = \theta$. Now let $k$ be the index where

$$u_k \geq u_i \quad \text{for all} \quad i = 1, 2, \ldots, n.$$ 

From the $k$-th row of $Au = \theta$, we obtain:

$$a_{k,1}u_1 + a_{k,2}u_2 + \cdots + a_{k,k-1}u_{k-1} + a_{k,k+1}u_{k+1} + a_{k,k+2}u_{k+2} + \cdots + a_{k,n}u_n = 0 - a_{k,k}u_k.$$ 

Hence

$$|a_{k,k}| \|u_k\| = \left| \sum_{i \neq k} a_{k,i} u_i \right| \leq \sum_{i \neq k} |a_{k,i}| \|u_i\| \leq \sum_{i \neq k} |a_{k,i}| \|u_k\| \leq |u_k| r_k(A),$$

which is a contradiction with the fact that $|a_{k,k}| > r_k(A)$.

**Gerschgorin’s Disk Theorem.** The eigenvalues of $A$ lie in the union of the disks

$$D_i(a_{ii}, r_i(A)), \quad \text{centered at} \quad a_{ii} \quad \text{with the radius} \quad r_i(A).$$

Moreover, If the union of $k$ disks is disjoint from the union of the other $n-k$ disks, then the former union contains exactly $k$ eigenvalues and the latter $n-k$ eigenvalues (counting multiplicities) of $A$.

**Proof.** We only prove the first part.

Let $\lambda$ be an eigenvalue of $A$ and define the singular matrix $B = A - \lambda I_n$. According to Levy-Desplanques theorem, at least one index $i$ must exist for which

$$|a_{i,i} - \lambda| = |b_{i,i}| \leq r_i(A).$$

The fact that $\sigma(A) = \sigma(A^t)$, we can obtain similar results by using columns of $A$ instead of its rows.

Each Gerschgorin disk uses data from one row of the matrix $A$. What if we consider two rows at a time? Can we find sets that contain eigenvalues of $A$, similarly to the Gerschgorin
theorem? This question was answered (affirmatively) by Alfred Brauer in 1947 when he came up with sets that he called the ovals of Cassini. We present the theorem without proof.

**Ovals of Cassini.** Let $A$ be an $n \times n$ matrix. For $1 \leq i, j \leq n$ with $i \neq j$, define the complex set

$$K_{i,j} = \{ z : |z - a_{i,i}| \times |z - a_{j,j}| \leq r_i(A) \times r_j(A) \}.$$ 

Then $\sigma(A) \subseteq \bigcup_{i \neq j=1}^{n} K_{i,j}$.

**Corollary 1.** If $|a_{i,i}| \times |a_{j,j}| > r_i(A) \times r_j(A)$ ($i = 1, 2, \ldots, n$ and $i \neq j$), then the matrix $A$ is invertible.

Are the ovals of Cassini better at bounding eigenvalues than Gerschgorin disks? The answer is yes! The ovals of Cassini are always at least as good as the Gerschgorin disks at estimating the eigenvalues of any square matrix.

**Example.** Consider the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 9 \end{pmatrix}.$$ 

Since $A$ is symmetric, then the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are real. By Gerschgorin’s Disk theorem, two eigenvalues are in $[0, 4] \cup [1, 5] = [0, 5]$ and the dominant eigenvalue is in $[7, 11]$. Is $A$ invertible?

According to the Ovals of Cassini, we have:

- $|z - 2| \times |z - 3| \leq 2 \times 2 = 4$,
- $|z - 2| \times |z - 9| \leq 2 \times 2 = 4$,
- $|z - 3| \times |z - 9| \leq 2 \times 2 = 4$.

From $|z - 2| \times |z - 3| \leq 4$, we conclude that $\lambda_1 > 0$; thus $A$ is invertible.

From $|z - 2| \times |z - 9| \leq 4$, we conclude that either $\lambda_1 \approx 2$ or $\lambda_3 \approx 9$.

From $|z - 3| \times |z - 9| \leq 4$, we conclude that either $\lambda_2 \approx 3$ or $\lambda_3 \approx 9$.

**Definition.** The Frobenius norm of an $m \times n$ matrix $A$ defined as the square root of the trace of the positive semi-definite matrix $A A^\ast$ which is also the sum of square of modulus of entries of $A$.

$$||A||_F = \sqrt{\text{trace}(A A^\ast)} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2}.$$ 

Although the Frobenius norm is multiplicative, but $||I_n|| = n \neq 1$.

**Schur’s Inequalities.**

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{i,j}|^2 = ||A||_F^2.$$ 

**Proof.** According to Schur’s lemma, $T = U^\ast A U$ for some upper triangular matrix $T = (t_{i,j})$ and unitary matrix $U = (u_{i,j})$. Thus

$$T^\ast = U^\ast A^\ast U \quad \text{and} \quad TT^\ast = (U^\ast A U) (U^\ast A^\ast U) = U^\ast A A^\ast U.$$
The facts that \( \text{trace} (AA^*) = \text{trace} (TT^*) \) and \( \text{trace} (AA^*) = \sum_{i,j=1}^{n} |a_{i,j}|^2 \) imply that

\[
\sum_{i,j=1}^{n} |a_{i,j}|^2 = \text{trace}(AA^*) = \text{trace}(TT^*) = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i=1}^{n} |t_{i,i}|^2 .
\]

Hence the desired conclusion.

**Companion matrix**

Consider now the polynomial of degree \( n \)

\[
P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n .
\]

The polynomial \( P \) is said to be **monic**, if the leading coefficient \( a_0 \) equals one. Clearly, the matrix

\[
Q(x) = \frac{1}{a_0} P(x) = x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n .
\]

is monic. To this monic polynomial we associate an \( n \times n \) matrix

\[
C_P = \begin{pmatrix}
-b_1 & -b_2 & -b_3 & \ldots & -b_{n-1} & -b_n \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
& \vdots & \ddots & \ddots & \vdots & \vdots \\
& \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 1 & 0
\end{pmatrix} .
\]

The matrix \( C_P \) is called the **Companion matrix** of \( P(x) \).

**Theorem 2.** The roots of the polynomial \( p(x) \) are the eigenvalues of the companion matrix \( C_p \).

**Corollary.** Consider a monic polynomial \( P(x) \) of degree \( n \). Then by using the Gerschgorin’s Disk Theorem, we obtain:

(i) all the roots of \( P(x) \) is contained within \( D_r \cap D_c \), where

\[
D_r = D(0,1) \cup D\left(-b_1, \sum_{k=0}^{n-2} |b_k| \right)
\]

\[
D_c = \left[D(-b_1,1) \cup D(0,1 + |b_2|) \cup \ldots \cup D(0,1 + |b_{n-2}|) \cup D(0,|b_{n-1}|) \right] ;
\]

(ii) if \( \{x_1,x_2,\ldots,x_n\} \) are the \( n \) roots of \( P(x) \), then

\[
\sum_{k=1}^{n} x_k = -b_1 .
\]

Note that by using ovals of Cassini, one may obtain a better domain.