Jordan normal form or Jordan canonical form (named in honor of Camille Jordan) shows that by changing the basis, a given square matrix \( A \) can be transformed into a certain normal form

\[
J = \begin{bmatrix}
J_1 & 0 & \cdots & 0 \\
0 & J_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & J_p
\end{bmatrix},
\]

where each block \( J_k \) is a square matrix of the form

\[
J_k = \begin{bmatrix}
\lambda_k & 1 & 0 & 0 \\
0 & \lambda_k & \ddots & 0 \\
0 & \ddots & \ddots & 1 \\
0 & 0 & 0 & \lambda_k
\end{bmatrix}.
\]

Note that \( \lambda_k \) is a repeated eigenvalue of \( A \).

**Definition 1.** Given an eigenvalue \( \lambda_k \) of an \( n \times n \) matrix \( A \):

a. The **algebraic multiplicity** of \( \lambda_k \) is the multiplicity of \( \lambda_k \) in the characteristic polynomial \( K_A(\lambda) \) of \( A \).

b. The **geometric multiplicity** of \( \lambda_k \) (which is the number of linearly independent eigenvectors associated with \( \lambda_k \)) is the dimension of \( \text{Ker}(A - \lambda_k I_n) \). Thus

\[
n - \text{rank}(A - \lambda_k I_n)
\]

is the geometric multiplicity of \( \lambda_k \) which is also the number of Jordan blocks corresponding to \( \lambda_k \).

- The orders of the Jordan Blocks of \( \lambda_k \) must sum to the algebraic multiplicity of \( \lambda_k \).
- The number of Jordan blocks corresponding to an eigenvalue \( \lambda_k \) is its geometric multiplicity.
- The matrix \( A \) is **diagonalizable** if and only if, for any eigenvalue \( \lambda \) of \( A \), its geometric and algebraic multiplicities coincide.

A non diagonalizable matrix is sometimes called a **defective matrix**.

**Theorem 1.** The order of the largest Jordan Block of \( A \) corresponding to an eigenvalue \( \lambda \) (called the **index** of \( \lambda \)) is the smallest value of \( k \in \mathbb{N} \) such that

\[
\text{rank}(A - \lambda I)^k = \text{rank}(A - \lambda I)^{k+1}.
\]

**Proof.** If \( \text{rank}(A - \lambda I)^k = \text{rank}(A - \lambda I)^{k+1} \), then

\[
0 = \text{dim}[\text{ker}(A - \lambda I)^{k+1}] - \text{dim}[\text{ker}(A - \lambda I)^k] = \text{the number of Jordan Blocks of order } \geq k+1.
\]
Since this is the smallest such $k$ which makes this equation equal to zero, it implies that

$$0 \leq \dim \left[ \ker (A - \lambda I)^k \right] - \dim \left[ \ker (A - \lambda I)^{k-1} \right] = \text{the number of Jordan Blocks of order } \geq k. \quad \square$$

**Definition 2.** Let $A$ be $n \times n$ matrix, then there exists a unique monic polynomial of least positive degree $m(x)$ such that $m(A) = Z_n$, where $Z_n$ is the zero matrix. The polynomial $m(x)$ is called the minimal polynomial of $A$. This polynomial is denoted by $m_A(\lambda)$.

**Theorem 2.** For any $n$ degree polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

there is an $b \times n$ matrix $A$ for which it is the minimal polynomial.

**Proof.** Consider the $n \times n$ matrix $A$ and the vector $e_k$:

$$A = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & -a_0 \\ 1 & 0 & \cdots & \cdots & \cdots & -a_1 \\ 0 & 1 & 0 & \cdots & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} \vdots \end{bmatrix}$$

Observe that

$$I_ne_1 = e_1 = A^0e_1, \quad Ae_1 = e_2 = Ae_1, \quad Ae_2 = e_3 = A^2e_1, \quad \vdots \quad Ae_{n-1} = e_n = A^{n-1}e_1.$$  

$$Ae_n = -a_{n-1}e_n - a_{n-2}e_{n-1} - \cdots - a_1e_2 - a_0e_1$$

Since $Ae_n = A^ne_1$, it follows that

$$p(A)e_1 = A^n e_1 + a_{n-1}A^{n-1}e_1 + a_{n-2}A^{n-2}e_1 + \cdots + a_1Ae_1 + a_0e_1 = Z_n.$$  

Also

$$p(A)e_k = p(A)A^{k-1}e_1 = A^{k-1}p(A)e_1 = A^{k-1}Z_n = Z_n \quad k = 2, 3, \ldots$$

Hence $p(A)e_j = Z_n$ for $j = 1, 2, 3, \ldots$ Thus $p(A)e_j = Z_n$.

Suppose now that for some $m < n$ there exist a polynomial

$$q(x) = x^m + b_{m-1}x^{m-1} + m_{m-2}x^{m-2} + \cdots + b_1x + b_0$$

such that $q(A) = Z_n$. Then

$$q(A)e_1 = A^me_1 + b_{m-1}A^{m-1}e_1 + b_{m-2}A^{m-2}e_1 + \cdots + b_1Ae_1 + b_0e_1 = Z_n.$$  

But the vectors $e_{m+1}, e_m, e_{m-1}, e_1$ are linear independent from which we conclude that $q(A) = Z_n$ is impossible. Thus $p(x)$ is minimal. \quad \square
• Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigenvalues of $A$, and $s_k$ be the index of $\lambda_k$. It is clear from the Jordan normal form that the minimal polynomial of $A$ has degree $\sum_{k=1}^{m} s_k$.

**Remark.** While the Jordan normal form determines the minimal polynomial, the converse is not true. This leads to the notion of **elementary divisors**. The elementary divisors of a square matrix $A$ are the characteristic polynomials of its Jordan blocks. The factors of the minimal polynomial $m_A(\lambda)$ are the elementary divisors of the largest degree corresponding to distinct eigenvalues.

Although the minimal polynomial of the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is the same as its characteristic polynomial, but $A$ is not diagonalizable; because the geometric multiplicity of $\lambda = 2$ is one which is different from its algebraic multiplicity which two.

While the Jordan normal form determines the minimal polynomial, the converse is not true. This leads to the notion of **elementary divisors**. The elementary divisors of a square matrix $A$ are the characteristic polynomials of its Jordan blocks. The factors of the minimal polynomial $m_A(\lambda)$ are the elementary divisors of the largest degree corresponding to distinct eigenvalues.

One of the methods for determining the minimal polynomial of a matrix (beside looking at its Jordan normal form) involves testing all possible factors.

One could use the Krylov’s method for finding the characteristic polynomial to see if the minimal polynomial has a lower degree than the characteristic polynomial. Here is how:

Choose an arbitrary $n$-dimensional nonzero column vector $v$ such as $e_1$; then define the matrix

$$V = [v, Av, A^2v, \ldots, A^{n-2}v, A^{n-1}v].$$

If the matrix $V$ has rank $n$, then both polynomials are equal, if not, then choose another initial vector. If after a few trials, you fail to find an invertible matrix, then start testing all possible factors.

Suppose the algebraic multiplicity of $\lambda_j$ is $r_j$. If for some $k \leq r_j$ (the smallest), the rank of $[A - \lambda I_n]^k$ becomes $n - r_j$, then the degree of the factor $(\lambda - \lambda_k)$ in $m_A(\lambda)$ equals $k$.

To obtain a Jordan normal form of an $n \times n$ diagonalizable matrix $A$, we use $n$ linearly independent eigenvectors to construct a matrix $P$, where the matrix

$$P^{-1}AP = J(A)$$

will be the Jordan normal form.

**Generalized Eigenvector.** If $\lambda$ is an eigenvalue of a defective $n \times n$ matrix $A$ with an algebraic multiplicity greater than its geometric multiplicity, then any nonzero vector $v$ satisfying:

$$(A - \lambda I_n)^k v = \theta, \quad \text{but} \quad (A - \lambda I_n)^{k-1} v \neq \theta, \quad \text{for} \quad k = 2, 3, \ldots$$

is called a generalized eigenvector.

**Definition 3.** Let $A$ be an $n \times n$ matrix and let $v$ be a generalized eigenvector of $A$ corresponding to the eigenvalue $\lambda$. Suppose $m$ is the smallest positive integer for which $(A - \lambda I_n)^m v = \theta$. Thus the order set

$$\{ (A - \lambda I_n)^{m-1} v, (A - \lambda I_n)^{m-2} v, \ldots, (A - \lambda I_n) v, v \}$$
is called a **cycle** of the generalized eigenvectors of $A$ corresponding to the eigenvalue $\lambda$. The vectors $(A - \lambda I)^{m-1}v$ and $v$ are called the **initial vector** and the **end vector** of the cycle respectively. We say that the **length** of the cycle is $m$.

- Every cycle of generalized eigenvector of $A$ is linearly independent.
- To construct the Jordan normal form of $A$, we form a sequence of generalized eigenvectors that satisfy:

  $$(A - \lambda I_n) v_1 = u_1, \quad (A - \lambda I_n) v_2 = v_1, \quad (A - \lambda I_n) v_3 = v_2, \ldots$$

The eigenvectors and generalized eigenvectors of $A$ form the columns of the invertible matrix $P$ which gives us the Jordan normal form $J(A)$.

Suppose the geometric multiplicity of $\lambda$ is $m$ and its algebraic multiplicity is larger than $m$.

To construct the Jordan norm of $A$, first we need to find the eigenvectors $u_1, u_2, \ldots, u_m$ with

$$Au_1 = \lambda u_1, \quad Au_2 = \lambda u_2, \ldots, \quad Au_m = \lambda u_m;$$

then we obtain $v_{kj}'s$, the generalized eigenvectors produced by $u_k's$. The matrix $P$ will be formed as follows:

$$P = \begin{bmatrix} u_1 & v_{11} & v_{12} & \ldots & v_{1r} & u_2 & v_{21} & v_{22} & \ldots & v_{2s} & u_3 & \ldots \end{bmatrix}.$$ 

If we are unable to produce generalized eigenvectors with $u_j's$, then we need to find other linearly independent eigenvectors.

We shall explain how to create the Jordan normal form of a matrix with a simple example. For the sake of simplicity, we choose the matrix $A$ as a matrix in Jordan normal form.

Consider the $9 \times 9$ matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$ 

Let $B = A - 2I_9$. Then rank $(B) = 5$. Thus $A$ has four $(9 - 5 = 4)$ linearly independent eigenvectors. They are:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$
The generalized eigenvectors associated with \( u_1 \) are

\[
v_{11} = \text{pinv}(B)u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\quad \text{and} \quad v_{12} = \text{pinv}(B^2)u_1 = \text{pinv}(B)v_{11} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Note that \( v_{13} = \text{pinv}(B^3)u_1 = \text{pinv}(B^2)v_{11} = \text{pinv}(B)v_{12} \) is the zero vector. Hence one of the Jordan blocks of \( J(A) \) is a \( 3 \times 3 \) matrix.

Similarly, we find that the generalized eigenvectors associated with \( u_2 \) are

\[
v_{21} = \text{pinv}(B)u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\quad \text{and} \quad v_{22} = \text{pinv}(B^2)u_2 = \text{pinv}(B)v_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Hence \( J(A) \) has another \( 3 \times 3 \) Jordan block. At this point we know that the index of \( \lambda = 2 \) is three; therefore the minimal polynomial of \( A \) is \( m_A(\lambda) = (A - \lambda I_9)^3 \).

Since \( A \) has four linearly independent eigenvectors, only one of the remaining eigenvectors may have a generalized eigenvector. The generalized eigenvectors associated with \( u_3 \) is

\[
v_{31} = \text{pinv}(B)u_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

At this point we may construct the Jordan normal form of \( A \).

**Example.** Consider the matrices

\[
A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},
\]
where 3 is the only eigenvalue of both matrices. The characteristic polynomials are

\[ K_A(\lambda) = K_B(\lambda) = (\lambda - 3)^3. \]

The ranks of the matrices \((A - 3I_3)\) and \((B - 3I_3)\) are two and one, respectively. The minimal polynomials are

\[ m_A(\lambda) = (\lambda - 3)^3 \quad \text{and} \quad m_B(\lambda) = (\lambda - 3)^2. \]

To construct the Jordan normal form of the matrix \(A\), we only need one eigenvector but two generalized eigenvectors associated with \(\lambda = 3\). But the Jordan normal form of \(B\) requires two eigenvectors and only one generalized eigenvector associated with \(\lambda = 3\).

**Exercise.** Find the minimal polynomials of the following matrices

\[
A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -1 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -4 & -1 & 0 \\ 3 & -9 & -4 & 2 & -1 \\ 1 & 5 & 4 & 1 & 4 \end{bmatrix}.
\]

Once you found the minimal polynomiala of the above matrices, display their Jordan normal form.

*Jordan Canonical Forms for Matrices with Repeated Eigenvalues.* If the matrix has no repeated roots, then it is diagonalizable and there is no interest to us.

Consider the following \(4 \times 4\) matrices:

\[
A = \begin{bmatrix} -30 & -12 & 19 & 27 \\ -10 & 1 & 5 & 8 \\ -6 & -2 & 7 & 5 \\ -43 & -15 & 24 & 38 \end{bmatrix}, \quad B = \begin{bmatrix} -12 & -6 & 9 & 13 \\ -1 & 4 & 0 & 1 \\ -6 & -2 & 7 & 5 \\ -16 & -6 & 9 & 17 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -25 & -10 & 16 & 23 \\ -20 & -3 & 11 & 16 \\ -11 & -4 & 10 & 9 \\ -38 & -13 & 21 & 34 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} -7 & -4 & 6 & 9 \\ -11 & 0 & 6 & 9 \\ -11 & -4 & 10 & 9 \\ -11 & -4 & 6 & 13 \end{bmatrix}.
\]

with the characteristic polynomials:

\[ K_A(\lambda) = K_B(\lambda) = K_C(\lambda) = K_D(\lambda) = \lambda^4 - 16\lambda^3 + 96\lambda^2 - 256\lambda + 256 = (\lambda - 4)^4. \]

Thus \(\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 4\).

**Case 1.** The geometric multiplicity of the matrix \(A\) is one, so there is only one Jordan block.

The rank of the matrix \(\hat{A} = A - 4I_4\) is 3.

The vector \(u(A) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\) which is the solution of the homogeneous system \(\hat{A}u = \theta\) is the only
linearly independent eigenvector of \( A \). The vectors

\[
v(A) = \begin{bmatrix} -1 \\ -2 \\ -3 \\ 0 \end{bmatrix}, \quad w(A) = \begin{bmatrix} 3 \\ 9 \\ 11 \\ 0 \end{bmatrix}, \quad \text{and} \quad z(A) = \begin{bmatrix} 0 \\ -16 \\ -27 \\ 12 \end{bmatrix}
\]

are the generalized eigenvectors of \( A \), obtain by solving

\[
\hat{A} v = u(A), \quad \hat{A} w = v(A), \quad \text{and} \quad \hat{A} z = w(A) \text{ respectively.}
\]

By constructing the matrix \( P = [ u(A) \ v(A) \ w(A) \ z(A) ] \) and then find its inverse.

\[
P = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 1 & -2 & 9 & -16 \\ 1 & -3 & 11 & -27 \\ 1 & 0 & 0 & 12 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -60 & -24 & 36 & 49 \\ -43 & -15 & 24 & 34 \\ 6 & 3 & -4 & -5 \\ 5 & 2 & -3 & -4 \end{bmatrix}.
\]

The Jordan normal form of \( A \) is as follows:

\[
J(A) = P^{-1} A P = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.
\]

Notice that the index of \( \lambda = 4 \) is four, therefore \( m_A(\lambda) = (\lambda - 4)^4 \).

**Case 2 a.** The geometric multiplicity of the matrix \( B \) is two, so there are two blocks in the Jordan normal matrix. Since the rank of the matrix

\[
\hat{B} = B - 4I_4 = \begin{bmatrix} -16 & -6 & 9 & 13 \\ -1 & 0 & 0 & 1 \\ -6 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \end{bmatrix}
\]

is 2, the vectors

\[
u_1(B) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u_2(B) = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}
\]

which are the solution of the homogeneous system \( \hat{B} u = \theta \) are linearly independent eigenvectors of \( B \). To obtain the Jordan normal form of \( B \), we need to obtain two generalized eigenvectors of \( B \) by solving the systems \( B v = u_1(B) \) and \( B v = u_2(B) \). The generalized eigenvectors are

\[
v_1(B) = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2(B) = \begin{bmatrix} -2 \\ 0 \\ -5 \\ 1 \end{bmatrix}.
\]

The Jordan normal form of \( B \) is obtain from the matrix \( Q = [ u_1(B) \ v_1(B) \ u_2(B) \ v_2(B) ] \)
and its inverse $Q^{-1}$:

$$Q = \begin{bmatrix} 1 & -1 & 0 & -2 \\ 1 & 1 & 3 & 0 \\ 1 & -1 & 2 & -5 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q^{-1} = \begin{bmatrix} -5 & -2 & 3 & 5 \\ -16 & -6 & 9 & 13 \\ 7 & 3 & -4 & -6 \\ 5 & 2 & -3 & -4 \end{bmatrix}.$$  

We have

$$J(B) = Q^{-1}BQ = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$  

Notice that the index of $\lambda = 4$ is two, therefore $m_B(\lambda) = (\lambda - 4)^2$.

**Case 2 b.** The geometric multiplicity of the matrix $C$ is also two, so there are two blocks in the Jordan normal form. Since the rank of the matrix

$$\hat{C} = C - 4I_4 = \begin{bmatrix} -29 & -10 & 16 & 23 \\ -20 & -7 & 11 & 16 \\ -11 & -4 & 6 & 9 \\ -38 & 13 & 21 & 30 \end{bmatrix}$$

is 2, the vectors

$$u_1(C) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u_2(C) = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 2 \end{bmatrix}$$

are linearly independent eigenvectors of $C$. By solving the system $\hat{C}v = u_1(C)$, we obtain two linearly independent generalized eigenvectors

$$v_{11}(C) = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v_{12}(B) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}.$$  

But the vectors $u_1(C)$, $u_2(C)$, $v_{11}(C)$ and $v_{12}(C)$ are linearly dependent; so we should use one of the two options:

1. Remove one of the generalized eigenvectors and solve the system $\hat{C}v = u_2(C)$.
2. Solve either $\hat{C}w = v_{11}(C)$ or $\hat{C}w = v_{12}(C)$.

The system $\hat{C}v = u_2(C)$ is inconsistent, so we use the second option. The system $\hat{C}w = v_{11}(C)$ is also inconsistent, but $\hat{C}w = v_{12}(C)$ has the vector $w_{12}(C) = \begin{bmatrix} -2 \\ 8 \\ 0 \\ 1 \end{bmatrix}$ as a solution.

Now by choosing $u_1(C)$, $u_2(C)$, $v_{12}(C)$ and $w_{12}(C)$, we may construct the matrix

$$R = [u_1(C) \ v_{12}(C) \ w_{12}(C) \ u_2(C)].$$
and find its inverse $R^{-1}$.

\[
R = \begin{bmatrix}
1 & 1 & -2 & 0 \\
1 & 0 & 8 & 3 \\
1 & -1 & 0 & -1 \\
1 & 2 & 1 & 2
\end{bmatrix}, \quad \text{and} \quad R^{-1} = \begin{bmatrix}
3 & 1 & -1 & -2 \\
-20 & -7 & 11 & 16 \\
-9 & -3 & 5 & 7 \\
23 & 8 & -13 & -18
\end{bmatrix}.
\]

The Jordan normal form of $C$ is as follows:

\[
J(C) = R^{-1}CR = \begin{bmatrix}
4 & 1 & 0 & 0 \\
0 & 4 & 1 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}.
\]

Notice that the index of $\lambda = 4$ is three, therefore $m_C(\lambda) = (\lambda - 4)^3$.

**Case 3.** The rank of the matrix

\[
\hat{D} = D - 4I_4 = \begin{bmatrix}
-11 & -4 & 6 & 9 \\
-11 & -4 & 6 & 9 \\
-11 & -4 & 6 & 9 \\
-11 & -4 & 6 & 9
\end{bmatrix}
\]

is one, so there are three blocks in the Jordan normal form. The three linearly independent eigenvectors of $D$ are as follows:

\[
u_1(D) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2(D) = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_3(D) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -2 \end{bmatrix}.
\]

The generalized eigenvector $v_1(D) = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is a solution of the system $\hat{D}v = u_1(D)$.

Note that the eigenvectors $\begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 4 \\ 0 \\ 3 \end{bmatrix}$ are linearly independent but do not produce any generalized eigenvector since all the rows of $\hat{D}$ are identical but components of these three vectors are not all equal.

The matrix $S = [u_1(D) \ v_1(D) \ u_2(D) \ u_3(D)]$ and its inverse:

\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & -1 & 3 & 0 \\
1 & 1 & 2 & 3 \\
1 & -1 & 0 & -2
\end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-11 & -4 & 6 & 9 \\
-4 & -1 & 2 & 3 \\
6 & 2 & -3 & -5
\end{bmatrix}.
\]
will produce the Jordan normal form as follows:

\[
J(D) = S^{-1}DS = \begin{bmatrix}
4 & 1 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}.
\]

Notice that the index of \( \lambda = 4 \) is two, therefore \( m_D(\lambda) = (\lambda - 4)^2 \).

**Remark.** The following vectors

\[
u_1(D) = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \quad u_2(D) = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_3(D) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -2 \end{bmatrix}
\]

are also linearly independent eigenvectors of the matrix \( D \); but none of them could produce a generalized eigenvector \( v \) which is needed to obtain \( J(D) \).

Since \( \hat{D}^2 \) is the zero matrix, it follows that any vector is a generalized eigenvector of \( D \); for example the first three columns of the invertible matrix

\[
T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
4 & 3 & 0 & 0 \\
3 & 2 & 3 & 0 \\
1 & 0 & -2 & 1
\end{bmatrix}
\]

are the eigenvectors of \( D \), and the forth column is a generalized eigenvector in the generalized sense. Unfortunately

\[
T^{-1}DT = \begin{bmatrix}
4 & 0 & 0 & 9 \\
0 & 4 & 0 & -9 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}
\]

is not the Jordan normal form of \( D \). In this case, we need to go back to our eigenvectors and choose the ones that will produce a Jordan normal form.

**Example.** Consider the matrix \( M = \begin{bmatrix} 7 & 4 & 3 & -7 \\
3 & 5 & 2 & -5 \\
1 & 0 & 2 & 0 \\
5 & 4 & 3 & -5 \end{bmatrix} \) with the characteristic polynomial

\[
K_M(\lambda) = \lambda^4 - 9\lambda^3 + 29\lambda^2 - 39\lambda + 18.
\]

The eigenvalues are \( \lambda_1 = 1, \lambda_2 = 2, \) and \( \lambda_3 = \lambda_4 = 3 \). with the corresponding eigenvectors:

\[
u_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad u_3 = u_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.
\]
The equation \((M - 3I_4) v = u_3\) produces the generalized eigenvector \(v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\).

Now we can construct the Jordan normal form \(J(M)\) of \(M\) with

\[
P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\]

with \(P^{-1} = \begin{bmatrix} -1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -2 \\ 2 & 2 & 1 & -3 \end{bmatrix}\).

Hence

\[
J(M) = P^{-1}MP = \begin{bmatrix} -1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -2 \\ 2 & 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 7 & 4 & 3 & -7 \\ 3 & 5 & 2 & -5 \\ 1 & 0 & 2 & 0 \\ -1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.
\]

**Numerical Analysis.** If the matrix \(A\) has multiple eigenvalues, or is close to a matrix with multiple eigenvalues, then its Jordan normal form is very sensitive to perturbations. Consider for instance the matrix \(A = \begin{bmatrix} 1 & 1 \\ \varepsilon & 1 \end{bmatrix}\). If \(\varepsilon = 0\), then the Jordan normal form is simply \(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\). However, for \(\varepsilon \neq 0\), the Jordan normal form is \(\begin{bmatrix} 1 + \sqrt{\varepsilon} & 0 \\ 0 & 1 - \sqrt{\varepsilon} \end{bmatrix}\).

This ill conditioning makes it very hard to develop a robust numerical algorithm for the Jordan normal form, as the result depends critically on whether two eigenvalues are deemed to be equal. For this reason, the Jordan normal form is usually avoided in numerical analysis.

**♣ System of Linear Differential Equations.** Consider the system

\[
X'(t) = AX(t) \quad \text{with} \quad P^{-1}AP = J(A).
\]

Let \(X(t) = PY(t)\) and \(Y'(t) = P^{-1}X'(t)\). Then

\[
Y'(t) = P^{-1}X'(t) = P^{-1}AX(t) = P^{-1}APY(t) = J(A)Y(t).
\]

Thus by solving \(Y'(t) = J(A)Y(t)\), we may obtain the solution of the original system as follows:

\[
X'(t) = PY'(t) = P[J(A)Y(t)] = P[PP^{-1}APY(t)] = PP^{-1}APY(t) = APY(t) = AX(t).
\]

Consider the following matrices.

\[
A = \begin{bmatrix} 5 & 9 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 4 & 8 \\ -1 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}.
\]
**Step 1.** The characteristic polynomial of $A$ and $B$ are as follows:

\[
K_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 5 - \lambda & 9 & -2 \\ -1 & -1 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = (2 - \lambda) \begin{vmatrix} 5 - \lambda & 9 \\ -1 & -1 - \lambda \end{vmatrix}
\]

\[
= (2 - \lambda) \left[ (5 - \lambda)(-1 - \lambda) + 9 \right] = (2 - \lambda)^3
\]

and

\[
K_B(\lambda) = \det(B - \lambda I_3) = \begin{vmatrix} 4 - \lambda & 4 & 8 \\ -1 & -\lambda & -4 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = (2 - \lambda) \begin{vmatrix} 4 - \lambda & 4 \\ -1 & -\lambda \end{vmatrix}
\]

\[
= (2 - \lambda) \left[ -\lambda(4 - \lambda) + 4 \right] = (2 - \lambda)^3.
\]

By setting $K_A(\lambda)$ to zero, we obtain $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Hence the multiplicity of 2 in the $K_A(\lambda)$ is 3. We obtain the same eigenvalues by setting $K_B(\lambda)$ to zero; the multiplicity of 2 in the $K_A(\lambda)$ is also 3.

**Step 2.** Next, we are going to see if $A$ and $B$ are diagonalizable. We have:

\[
A - 2I_3 = \begin{bmatrix} 3 & 9 & -2 \\ -1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B - 2I_3 = \begin{bmatrix} 2 & 4 & 8 \\ -1 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The nullity of $A - 2I_3$ is 1; but the nullity of $B - 2I_3$ is 2. Hence there is one eigenvector for $A$, corresponding to the eigenvalue $\lambda = 2$; but two linearly independent eigenvectors for $B$, corresponding to the eigenvalue $\lambda = 2$. Therefore neither $A$ nor $B$ are diagonalizable.

**Step 3.** To find the eigenvectors of $A$ and $B$, we need to solve the following homogeneous linear systems:

\[
[A - 2I_3 | 0] = \begin{bmatrix} 3 & 9 & -2 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

and

\[
[B - 2I_3 | 0] = \begin{bmatrix} 2 & 4 & 8 & 0 \\ -1 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

By using some row operations, we obtain:

\[
[A - 2I_3 | 0] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

and

\[
[B - 2I_3 | 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

By solving the above systems, we find

\[
u(A) = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \quad u_1(B) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_2(B) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.
\]

Notice that $u_1(B)$ and $u_2(B)$ are linearly independent.

**Step 4.** Next, we need to find generalized eigenvectors for $A$ and $B$, corresponding to the eigenvalue $\lambda = 2$. From

\[
[A - 2I_3 | u(A)] = \begin{bmatrix} 3 & 9 & -2 & 3 \\ -1 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]
we obtain the linearly independent generalized eigenvectors:

\[ v_1(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } v_2(A) = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}. \]

But the matrix \( P = [u(A), v_1(A), v_2(A)] = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \) is singular, so we need to find a second generation generalized eigenvector. For this, we solve the following system, using \( v_1(A) \) (we could also use \( v_2(A) \)):

\[
\begin{bmatrix}
A - 2 I_3 & | & v_1(A)
\end{bmatrix} = \begin{bmatrix}
3 & 9 & -2 & | & 1 \\
-1 & -3 & 1 & | & 0 \\
0 & 0 & 0 & | & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 1 & | & 1 \\
-1 & -3 & 1 & | & 0 \\
0 & 0 & 0 & | & 0
\end{bmatrix},
\]

the vector \( w_{11} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) is a solution and

\[
Q = [u(A), v_1(A), w_{11}(A)] = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } Q^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}.
\]

The Jordan normal form of \( A \) is

\[
J(A) = Q^{-1}AQ = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.
\]

To find the Jordan normal form of \( B \), we only need to find one generalized eigenvector; using either \( u_1(B) \) or \( u_2(B) \); we just choose \( u_1(B) \). From

\[
\begin{bmatrix}
B - 2 I_3 & | & u(A)
\end{bmatrix} = \begin{bmatrix}
2 & 4 & 8 & 2 \\
-1 & -2 & -4 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 & 0 \\
-1 & -2 & -4 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

we obtain the linearly independent generalized eigenvector \( v_{11}(B) = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \).

Note that \( v_{12}(B) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) is another generalized eigenvector, but does not produce a non-singular matrix. The matrix

\[
R = [u_1(B), v_{11}(B), u_2(B)] = \begin{bmatrix} 2 & -1 & 4 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}
\]
has an inverse \( R^{-1} = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix} \). The Jordan normal form of \( A \) is

\[
J(A) = R^{-1}AR = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.
\]

**Exercise.** Consider, the following matrices:

\[
A = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.
\]

with Characteristic Polynomials:

\[
K_A(\lambda) = K_B(\lambda) = K_C(\lambda) = K_D(\lambda) = (\lambda - 2)^4.
\]

(i) Find the Jordan normal forms \( J(A), J(B), J(C), \) and \( J(D) \) and invertible matrices

\[
P_A, \quad P_B, \quad P_C, \quad \text{and} \quad P_D;
\]

such that

\[
P_A^{-1}AP_A = J(A), \quad P_B^{-1}BP_B = J(B), \quad P_C^{-1}CP_C = J(C), \quad \text{and} \quad P_D^{-1}DP_D = J(D).
\]

(ii) Let

\[
X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.
\]

Solve the following systems of linear differential equations:

\[
X'(t) = AX(t), \quad X'(t) = BX(t), \quad X'(t) = CX(t), \quad \text{and} \quad X'(t) = DX(t).
\]

Consider the following matrices

\[
A = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 5 & -8 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 1 & 4 \\ -4 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix}.
\]

Then find the Jordan normal form of \( A \) and \( B \).