Roots of Polynomials

 Bounds for the Roots of Polynomials. Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. If \( Au = \lambda u \), then \( \lambda \) and \( u \) are called the eigenvalue and eigenvector of \( A \), respectively. The eigenvalues of \( A \) are the roots of the characteristic polynomial
\[
K_A(\lambda) = \det (\lambda I_n - A).
\]
The eigenvectors are the solutions to the Homogeneous system
\[
(\lambda I_n - A) X = \theta.
\]
If \( A \) is symmetric, i.e., \( A^t = A \), then all the eigenvalues of \( A \) are real. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of \( A \), then
\[
\text{trace } (A) = \sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} a_{ii} \quad \text{and} \quad \det A = \prod_{k=1}^{n} \lambda_k.
\]
Our first theorem is known as the Gerschgorin’s Disks Theorem:

Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. For \( j = 1, 2, \ldots, n \), define
\[
r_j = \left( \sum_{i=1}^{n} |a_{ij}| \right) - |a_{jj}|.
\]
Let \( D_j(a_{jj}, r_j) \) be the disk of radius \( r_j \) with the center at the point \((0, a_{jj})\) of the complex plane. Then all the eigenvalues of the matrix \( A \) are contained within the union of the \( D_j \)’s. Thus
\[
D(A) = \bigcup_{j=1}^{n} D_j
\]
contains all the eigenvalues of \( A \).

Remark. Since \( A \) and \( A^t \) have the same set of eigenvalues, we may use the Gerschgorin’s Disks Theorem, for both \( A \) and \( A^t \) and get the best neighborhood \( D(A) \cap D(A^t) \) for the eigenvalues of \( A \).
Companion matrix
Consider now the polynomial of degree \( n \)
\[
P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.
\]
The polynomial \( P \) is said to be monic, if the leading coefficient \( a_0 \) equals one. Clearly, the matrix
\[
Q(x) = \frac{1}{a_0} P(x) = x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n.
\]
is monic. To this monic polynomial we associate an \( n \times n \) matrix
\[
C_P = \begin{pmatrix}
-b_1 & -b_2 & -b_3 & \ldots & -b_{n-1} & -b_n \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \ldots & 1 & 0
\end{pmatrix}.
\]
The matrix \( C_P \) is called the Companion matrix of \( P(x) \).

Theorem 1. The roots of the polynomial \( p(x) \) are the eigenvalues of the companion matrix \( C_P \).

Corollary. Consider a monic polynomial \( P(x) \) of degree \( n \). Then by using the Gerschgorin’s Disk Theorem, we obtain:

(i) all the roots of \( P(x) \) is contained within \( D_r \cap D_c \), where
\[
D_r = \left[ D(0,1) \cup D\left(-b_1, \sum_{k=0}^{n-2} |b_k| \right) \right] \quad \text{and}
\]
\[
D_c = \left[ D(-b_1,1) \cup D(0,1 + |b_2|) \cup \ldots \cup D(0,1 + |b_{n-2}|) \cup D(0,|b_{n-1}|) \right];
\]
(ii) if \( \{x_1, x_2, \ldots, x_n\} \) are the \( n \) roots of \( P(x) \), then
\[
\sum_{k=1}^{n} x_k = -b_1.
\]

♣ Rational Roots. Although a real polynomial may have complex roots, but there is a well known theorem concerning the rational roots of polynomial with integer coefficients.

Theorem 2. Let \( P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \) be a polynomial with integer coefficients. If \( p/q \) is a rational root of \( P(x) \), then \( a_n = pr \) and \( a_0 = qs \).

♣ Nested Form. Consider the following polynomial of degree \( n \)
\[
P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.
\]
The following form of $P(x)$ is called the nested form of $P(x)$:

$$P(x) = (((\cdots (((a_0)x + a_1)x + a_2)x \cdots )x + a_{n-1})x + a_n).$$

Finally, we present a root finding tool known as Horner’s method or Synthetic division.

**Synthetic Division.** Consider the polynomial:

$$P(x) = 2x^4 - 3x^2 + 3x - 4 = (((((2)x + 0)x - 3)x + 3)x - 4) = -4 + x(3 + x(-3 + x(0 + x(2))))).$$

The following chart shows how to evaluate $P(a), P'(a), P''(a), \ldots$ for $a = -2$.

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