A polynomial is an expression consisting of variables and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents. An example of a polynomial of a single variable, \( x \), is \( P(x) = x^2 - 3x + 5 \), which is a quadratic polynomial. An example in three variables is \( P(x, y, z) = x^2 y^3 + 2xyz^2 - 3y^3z + 4 \).

Polynomials appear in a wide variety of areas of mathematics and science. For example, they are used to form polynomial equations, which encode a wide range of problems, from elementary word problems to complicated problems in the sciences; they are used in calculus and numerical analysis to approximate other functions.

A polynomial \( P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \) of degree \( n \) has exactly \( n \) roots, where some roots may be complex or multiple roots. If \( z = a + ib \) is a complex root of the real polynomial \( P(x) \), then its conjugate, \( z = a - ib \) is also a root.

**Examples.**

1. \( P(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2) \) is a polynomial of degree three with \( x_1 = 0, x_2 = 1, \) and \( x_3 = 2 \) as its zeros.

2. \( Q(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3 = (x-1)(x-1)(x-1) \) is a polynomial of degree three with \( x_1 = 1 \), as its only zero. That means that \( x = 1 \) is a multiple (triple) root of \( Q(x) \).

3. \( R(x) = x^4 - 1 = (x-1)(x+1)(x+i)(x-i) \) is a polynomial of degree four with two real zeros and two complex zeros:

   \[ x_1 = -1, \quad x_2 = 1, \quad x_3 = -i, \quad \text{and} \quad x_4 = i. \]

**♣ Nested Form.**

Consider the following polynomial of degree \( n \)

\[ P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n. \]

To avoid using power, we use the following form of \( P(x) \) which is called the nested form of \( P(x) \):

\[ P(x) = (((a_0)x + a_1)x + a_2)x + \cdots + a_{n-1}x + a_n. \]

Notice that only “addition, subtraction, and multiplication” are used. Next, we present a root finding tool known as *Horner’s method* or *Synthetic division* which is derived from the nested form of polynomials.
Synthetic Division. The algorithm is used to evaluate \( P(a) \), \( P'(a) \), \( P''(a) \), \ldots, where \( P(x) \) is a polynomial.
Consider the polynomial:

\[
P(x) = 2x^4 - 3x^2 + 3x - 4 = (((((2)x + 0)x - 3)x + 3)x - 4).
\]

The following chart shows how to evaluate \( P(a) \) for \( a = -2 \).

| -2 | 2 0 -3 3 -4 |
|----|------|------|------|------|
| \( \times \) | \( \uparrow \) | \( \uparrow \) | \( \uparrow \) | \( \uparrow \) |
| 0  | -4  | 8   | -10  | 14   |

\[
P(-2) = 10
\]

Thus

\[
P(x) = 2x^4 - 3x^2 + 3x - 4 = (x + 2)(2x^3 - 4x^2 + 5x - 7) + 10.
\]

We may use Horner’s method to find \( P(a) = P^{(0)}(a) \), \( P'(a) = P^{(1)}(a) \), \ldots, \( P^{(k)}(a) \), \ldots

| -2 | 2 0 -3 3 -4 |
|----|------|------|------|------|
| 0  | -4  | 8   | -10  | 14   |

\[
10 \times 0! = p^{(0)}(-2)
\]

| -2 | 2 -4 5 -7 |
|----|------|------|------|------|
| 0  | -4  | 16   | -42 |

\[
-49 \times 1! = p^{(1)}(-2)
\]

| -2 | 2 -8 21 |
|----|------|------|
| 0  | -4  | 24   |

\[
45 \times 2! = p^{(2)}(-2)
\]

We may use Horner’s method to find the limit of a rational function.

Example. Find

\[
\lim_{x \to 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - 4x^2 + 5x - 2}.
\]
Thus \[ \lim_{x \to 1} \frac{x^3 - 3 x^2 + 3 x - 1}{x^3 - 4 x^2 + 5 x - 2} = \lim_{x \to 1} \frac{x^2 - 2 x + 1}{x^2 - 3 x + 2} = \lim_{x \to 1} \frac{x - 1}{x - 2} = \lim_{x \to 1} \frac{0}{-1} = 0. \]

Note that in the above example, we have

\[ \lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{f''(x)}{g''(x)} = \frac{f''(1)}{g''(1)} = \frac{0}{-1} = 0. \]

We are actually using Horner’s method instead of L’Hospital Rule.

\( \blacklozenge \) Rational Zeros Theorem. If \( P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \) is a polynomial with integer coefficients, then every rational zero of \( P \) is of the form \( \frac{p}{q} \), where \( p \) is a factor of the constant coefficient \( a_n \) and \( q \) is a factor of the leading coefficient \( a_0 \).

**Example.** Let \( P(x) = x^4 - 5 x^3 - 5 x^2 + 23 x + 10 \). The leading coefficient \( a_0 = 1 \), so if \( P(x) \) has some rational zeros, they must be integers. Also they are divisors of the constant term \( a_5 = 10 \). Thus the possible candidates are:

\[ \pm 1, \quad \pm 2, \quad \pm 5, \quad \pm 10. \]

The zeros of the polynomial \( x^2 - 2 x - 1 \) are \( 1 \pm \sqrt{2} \). Hence \( P(x) \) has two integer zeros and two irrational zeros; they are:

\[ 5, \quad -2, \quad (1 - \sqrt{2}), \quad \text{and} \quad (1 + \sqrt{2}). \]
\section*{Products of Polynomials.}

The polynomial

\[ P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n. \]

can be expressed as the vector

\[(a_0, a_1, a_2, \ldots, a_{n-1}, a_n)\]

We shall explain how to multiply two polynomials by using vector multiplication with the use of an example.

\textbf{Example.} Let \( P(x) = x^4 - 5 x^3 - 5 x^2 + 23 x + 10 \) and \( Q(x) = x^3 - 4 x + 2 \).

Then \( P(x) Q(x) \) may be obtain as follows:

\[
\begin{array}{cccccc}
1 & -5 & -5 & 23 & 10 & \times \\
1 & 0 & -4 & 2 & & \\
2 & -10 & -10 & 46 & 20 & \\
-4 & 20 & 20 & -92 & -40 & \\
1 & -5 & -5 & 23 & 10 & \\
1 & -5 & -9 & 45 & 20 & -102 & 6 & 20 \\
x^7 - 5 x^6 - 9 x^5 + 45 x^4 + 20 x^3 - 102 x^2 + 6 x + 20 & \\
\end{array}
\]

Notice that vector multiplication is simpler than regular multiplication, since it does not involve the carry over operation.

\section*{Long Division of Polynomials.}

The long division may be simplified with the use of vectors. We shall explain the vector division by using an example.

Let \( P(x) = x^5 - 5 x^3 - 5 x^2 + 23 x + 10 \) and \( Q(x) = x^2 - 4 x + 2 \).

Then

\[ P(x) \div Q(x) = D(x) + R(x) \]

\( D(x) \) and the remainder \( R(x) \) may be obtain as follows:

\[
\begin{array}{cccccc}
1 & 4 & 9 & 23 & & \\
1 & 0 & -5 & -5 & 23 & 10 & \\
\pm1 & \mp4 & \mp2 & & & \\
0 & 4 & -7 & -5 & 23 & 10 & \\
\pm4 & \mp16 & \pm8 & & & \\
0 & 9 & -13 & 23 & 10 & \\
\pm9 & \mp36 & \pm18 & & & \\
0 & 23 & 5 & 10 & \\
\pm23 & \mp92 & \pm46 & & & \\
0 & 97 & -36 & & & \\
\end{array}
\]
Thus

\[ P(x) = Q(x)(x^3 + 4x^2 + 9x + 23) + (97x - 36). \]

Note that ± and ∓ indicate change of signs. When 4 is multiplied by (1, −4, 2), the result is (4, −16, 8). Like the regular division, this vector must be subtracted from (4, −7, −5, 23, 10), so we change the sign of the vector (4, −16, 8) add it to (4, −7, −5, 23, 10).

♠ MacLauren Series.

Find the first five terms of the MacLauren series of the rational function

\[ f(x) = \frac{x^2 + 2x + 1}{x^3 + 3x^2 + 1}. \]

**Solution.** First we write both polynomials in ascending order:

\[ f(x) = \frac{1 + 2x + 3x^2}{1 + 3x^2 + x^3}. \]

Then we use the long division in the reverse order, as follows:

\[
\begin{array}{c|ccc}
 1 + 2x - 7x^3 & 1 & + 2x & + 3x^2 \\
 1 + 3x^2 + x^3 & 1 & ± 3x^2 & ± x^3 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
 0 & 2x & - x^3 \\
 1 + 3x^2 + x^3 & 0 & ± 6x^3 & ± 2x^4 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
 0 & - 2x^4 & 21x^5 & 7x^6 \\
 1 + 3x^2 + x^3 & 0 & ± 2x^4 & ± 6x^6 & ± 2x^7 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
 0 & 21x^5 & 13x^6 & 2x^7 \\
 1 + 3x^2 + x^3 & 0 & ± 2x^4 & ± 6x^6 & ± 2x^7 \\
\end{array}
\]

... ... ... ...

♠ Taylor Series.

Find the first five terms of the Taylor series about \( a = 2 \) of the rational function

\[ f(x) = \frac{3x^2 - 10x + 9}{x^3 - 3x^2 + 5}. \]

**Solution.** Let \( z = x - 2 \), then we replace \( x \) with \( z + 2 \). By expressing \( f(x) \) with respect to the variable \( z \), we obtain:

\[ f(x) = \frac{3(z + 2)^2 - 10(z + 2) + 9}{(z + 2)^3 - 3(z + 2)^2 + 5} = \frac{1 + 2z + 3z^2}{1 + z^2 + z^3}. \]
By using the above division for the MacLauren series, we obtain:

\[ f(x) = 1 + 2z - 7z^3 - 2z^4 + 21z^5 + \cdots \]

Hence

\[ f(x) = 1 + 2(x - 2) - 7(x - 2)^3 - 2(x - 2)^4 + 21(x - 2)^5 + \cdots \]